CMSC 250: Discrete Structures

Summer 2017

Lecture 7 - Outline

June 9, 2017

Functions and an Introduction to Counting

Inverse Functions

Let $f : A \to B$ be a bijection. The **inverse** function of f, denoted f^{-1}, is the function that maps an element $b \in B$ to the unique element $a \in A$ such that $f(a) = b$. Hence $f^{-1}(b) = a$ when $f(a) = b$.

Note that if f is not bijective then its inverse does not exist.

Function Composition

Let $f : A \to B$ and $g : B \to C$ be functions. The **composition** of the function g with f is the function $g \circ f : A \to C$, defined by

$$\forall x \in A, (g \circ f)(x) = g(f(x))$$

Problem: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Let $g : A \to A$ be such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let $f : A \to B$ be such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is $f \circ g$ and $g \circ f$?

Solution: The composition function $f \circ g$ is as follows: $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$(g \circ f)$ is not defined as the range of f is not a subset of the domain of g.

Problem: Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(x) = 2x + 3$. Let $g : \mathbb{Z} \to \mathbb{Z}$ be $g(x) = 3x + 2$. What is the composition of f with g? What is the composition of g with f?

Solution: $(f \circ g)(x) = f(g(x)) = 2(3x + 2) + 3 = 6x + 7$. Similarly, $(g \circ f)(x) = g(f(x)) = 3(2x + 3) + 2 = 6x + 11$. This example shows that commutative law does not apply to the composition of functions.

Prove: Let $f : A \to B$ and $g : B \to C$ be two functions. Prove that:

i. if f and g are surjective then so is $g \circ f$.

ii. if f and g are injective then so is $g \circ f$.

iii. if f and g are bijective then so is $g \circ f$.

Solution:

i. Let $c \in C$ be arbitrary. Since g is surjective, there must be a $b \in B$ such that $g(b) = c$. Since f is surjective, there must be a $a \in A$ such that $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$. This proves that $g \circ f$ is surjective.

ii. Let $a, a' \in A$ be arbitrary elements such that $(g \circ f)(a) = (g \circ f)(a')$. This means that $g(f(a)) = g(f(a'))$. Since g is injective we have $f(a) = f(a')$. Then since f is injective, we have $a = a'$.

iii. The bijectivity of $(g \circ f)$ follows from the injectivity and surjectivity of $(g \circ f)$.

Injection, Surjection and Bijection Rule

The Injection Rule

Let A and B be two finite sets. If there is an injective function from A to B, then $|A| \leq |B|$.

We can see this as follows. Since each element in A is mapped to a distinct element in B, this means that $|A| = |Ran(f)|$. Further, since $Ran(f) \subseteq B$, we know that $|Ran(f)| \leq |B|$. Therefore, $|A| \leq |B|$.

Note that we can further show that $|A| < |B|$ if we can show that there is some element $b \in B$ that is not mapped onto by the injective function.

The Surjection Rule

Let A and B be two finite sets. If there is a surjective function from A to B, then $|A| \geq |B|$.

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but $|A| < |B|$. Note that since each element in A is mapped to exactly one element in B, it must be that $|A| \geq |Ran(f)|$. Since $|Ran(f)| \leq |A|$ and $|A| < |B|$, we have that $|Ran(f)| < |B|$. Since $Ran(f) \subseteq B$ and $|Ran(f)| < |B|$, it must be that $Ran(f) \subset B$. Therefore, we have that $B \setminus Ran(f) \neq \varnothing$. In other words, there is an element in B such that it is not mapped onto by the function f. This contradicts the assumption that f is surjective.

Note that we can further show that $|A| > |B|$ if we can show that there is some element $b \in B$ that is mapped onto by two elements in A by the surjective function..

The Bijection Rule

Let A and B be two finite sets. If there is a bijective function from A to B, then $|A| = |B|$.

We can see this as follows. Since a bijective function is one that is injective and surjective, we can apply both the injection and surjection rules. Therefore, we know that $|A| \geq |B|$ and $|A| \leq |B|$. This is only possible when $|A| = |B|$.

What is very cool about these rules is that it allows us to determine the relative magnitudes of two different sets, even if we do not know how to explicitly count them yet.

Problem: Prove that there are more strings of length n made from the English alphabet with repeat than strings of length n made from the English alphabet without repeat. You may assume that $2 \leq n \leq 26$.

Solution: Let A be the set of all strings of length n made from the English alphabet with repeat. Let B be the set of all strings of length n made from the English alphabet without repeat. We want to show that $|A| > |B|$. We do so by constructing an injective function from B to A, and then showing that there is one element $a \in A$ that is not mapped onto.

Consider the following function $f : B \to A$, where f maps each element in B onto its identical element in A. We note that f is a function, since each element in B is mapped to exactly one element in A.

We also note that f is injective. Let $x, y \in B$ be arbitrary strings of length n made from English letters without repeat. Suppose $x \neq y$. Since $f(x) = x \neq y = f(y)$, we have that $f(x) \neq f(y)$.

By the injection rule, we have that $|B| \leq |A|$. We now show that $|B| < |A|$. Consider the string of the letter a repeated n times. This element is in A, but is not mapped onto by any element in B by the definition of f. Hence, we have that $|B| < |A|$.

Problem: Prove that there are more squares on an 8×8 chessboard than a 4×16 chessboard.

Solution: Let A be the set of all squares on the 8×8 chessboard, and B be the set of all squares on the 4×16 chessboard. We wish to show that $|A| > |B|$.

Let us consider the following function $f : B \to A$. For squares entirely within the top 4×8 section of the 4×16 board, the function maps them onto the same position on the 4×8 section of the left hand side of the 8×8 board. For squares entirely within the bottom 4×8 section of the 4×16 board, the function maps them onto the same position on the 4×8 section of the right hand side of the 8×8 board.

The last type of square to consider are those that are both on the top and bottom 4×8 sections of the 4×16 board. Note that any such overlapping square can only be in the center 4×6 region of the 4×16 board. For any squares in this section that overlap the two halves of the board, let the function map them to an equivalent (but rotated) 6×4 region in the middle of the 8×8 board, such that the horizontal line on the 4×16 board is lined up with the vertical line on the 8×8 board.

We note that f is clearly a function, since it maps squares in B to exactly one square in A.

Let us now show that f is injective. Let x and y be two arbitrary squares in B, such that $x \neq y$. We want to show that $f(x) \neq f(y)$.

There are a few cases here:

- **Case 1: x and y are on the same half of the 4 $\times$ 16 board**

  Without loss of generality, let x and y both be on the top half of the 4×16 board. Note that since we map x and y to their respective position in the 4×8 section on the left of the 8×8 board, we have that $f(x) \neq f(y)$, since $x \neq y$.

- **Case 2: x and y are on different halves of the 4 $\times$ 16 board**

  Without loss of generality, let x be on the top half on the board, and y be on the bottom half of the board. Since $f(x)$ is on the left half of the 8×8 board, and $f(y)$ is on the right half of the 8×8 board, we have that $f(x) \neq f(y)$.

- **Case 3: exactly one of x and y overlap between the top and bottom halves ofthe 4 $\times$ 16 board**

  Without loss of generality, let x be the overlapping square. Again without loss of generality, let y be on the top half of the 4×16 board.

  Since x is an overlapping square, $f(x)$ must be a square that overlaps the center vertical line that divides that left and right boards in the 8×8 board.

  Since y is not an overlapping square and is entirely in the top half of the 4×16 board, $f(y)$ must be entirely within the left half of the 8×8 board.

  Since $f(x)$ overlaps and $f(y)$ does not, it must be that $f(x) \neq f(y)$.

- **Case 4: both of x and y overlap between the top and bottom halves ofthe 4 $\times$ 16 board**

  Note that since we map x and y to their respective position in the 6×4 section in the middle of the 8×8 board, we have that $f(x) \neq f(y)$, since $x \neq y$.

Since we have shown that there is an injective function $f : B \to A$, we have that $|B| \leq |A|$.

Next, we wish to show that $|B| < |A|$ by showing that there is an element in A that is not mapped by f. One such element is the 8×8 square. This clearly is not mapped to by f, since there are no 8×8 squares in B.

Counting

Counting is a part of combinatorics, an area of mathematics which is concerned with the arrangement of objects of a set into patterns that satisfy certain constraints. We will mainly be interested in the number of ways of obtaining an arrangement, if it exists.

In this class, we will often refer to such arrangements as outcomes. This keeps us in line with the standard terminology used when we go ahead to study probability.

Outcomes and Outcome Spaces

An **outcome** is one possible arrangement that satisfies the constraints given. These outcomes are normally denoted with the Greek letter ω.

An **outcome space** is a set containing all possible outcomes. The outcome space is normally denoted Ω.

Considering the outcomes of a problem can often help to open the problem up. We will use the concept of outcomes throughout counting and probability.

Multiplication Rule. If the process of constructing outcomes can be broken down into k steps, then the total number of such outcomes can be counted as follows. Suppose:

- the first step can be performed in n_1 ways,
- the second step can be performed in n_2 ways, regardless of how the first step was performed,

$\vdots$

- the k^{th} step can be performed in n_k ways, regardless of how the preceding steps were performed, then

then the total number of outcomes that can be constructed is $n_1 \cdot n_2 \cdots n_k$.

To apply the multiplication rule think of outcomes that you are trying to count as the output of a multi-step operation. The possible ways to perform a step may depend on how the preceding steps were performed, but the number of ways to perform each step must be constant regardless of the action taken in prior steps.

Problem: A local deli that serves sandwiches offers a choice of three kinds of bread and five kinds of filling. How many different kinds of sandwiches are available?

Solution: Let us consider what an outcome looks like for this problem. One way of defining an outcome would be an ordered pair of the format: (Type of bread, Type of filling). One example outcome in this format would be (Whole Wheat, Chicken).

With this definition of an outcome, the outcome space $\Omega = \{$(Whole Wheat, Chicken), (Plain, Ham), (Whole Wheat, Ham), $\dots\}$.

We wish to find $|\Omega|$.

Let us think of how we construct an outcome, i.e. a sandwich, for this problem. We propose the following steps:

Step 1. Choose the bread - 3 ways.

Step 2. Choose the filling - 5 ways.

Step 1 can be done in 3 ways and Step 2 can be done in 5 ways. From the multiplication rule it follows that the number of available sandwich offerings is $3 \times 5 = 15$.

Example. Three officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Alex, Bob, Cat, and Dan. Suppose that for various reasons, Alex cannot be the president and either Cat or Dan must be the secretary. In how many ways can the officers be chosen?

Solution. Attempt 1. A set of three officers can be formed as follows.

Step 1. Choose the president.

Step 2. Choose the treasurer.

Step 3. Choose the secretary.

There are 3 ways to do Step 1. There are 3 ways of doing Step 2 (all except the person chosen in Step 1), and 2 ways of doing Step 3 (Cat or Dan). By multiplication rule, the number of different ways of choosing the officers is $3 \times 3 \times 2 = 18$.

The above solution is incorrect because the number of ways of doing Step 3 depends upon the outcome of Steps 1 and 2 and hence the multiplication rule cannot be applied. For example, if Cat was chosen to be the president in Step 1, and Alex was chosen to be the treasurer in Step 2, then there would be only one way to choose the secretary (it must be Dan)!

Attempt 2. A set of three officers can be formed as follows.

Step 1. Choose the secretary.

Step 2. Choose the president.

Step 3. Choose the treasurer.

Step 1 can be done in 2 ways (Cat or Dan). Step 2 can be done in 2 ways (Alex cannot be the president and the person chosen in Step 1 cannot be the president). Step 3 can be done in 2 ways (either of the two remaining people can be the treasurer). By multiplication rule, the numberof ways in which the officers can be chosen is $2 \times 2 \times 2 = 8$.

From the previous example we learn that there may not be a fixed order in which the operations have to be performed, and by changing the order a problem may be more readily solved by the multiplication rule. A rule of thumb to keep in mind is to *make the most restrictive choice first.*

Catalog of LaTeXCommands

$g \circ f$ - `g \circ f` | f^{-1} - `f^{-1}` | ω - `\omega` | Ω - `\Omega`