

# Lecture 8 - Outline

June 12, 2017

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## Counting

**Problem:** Let  $A$  and  $B$  be arbitrary sets. What is  $|A \times B|$ ?

**Solution:**

Let  $A = \{a_1, a_2, \dots, a_{|A|}\}$  and  $B = \{b_1, b_2, \dots, b_{|B|}\}$ .

Let us consider what an outcome looks like for this problem. Let the outcome be an ordered pair (Element from  $A$ , Element from  $B$ ). For example, one possible outcome would be  $(a_1, b_3)$ .

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose the first element of the pair from  $A$  -  $|A|$  ways

Step 2. Choose the second element of the pair from  $B$  -  $|B|$  ways

By the multiplication rule, we have that  $|\Omega| = |A \times B| = |A| \times |B|$ .

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**Problem:** Let  $A$  be an arbitrary set. What is  $|2^A|$ ?

**Solution:** Let  $A = \{a_1, a_2, \dots, a_{|A|}\}$ .

Let us consider what an outcome looks like for this problem. Let the outcome be the normal listing of a subset of  $A$ . For example, possible outcomes would be  $\emptyset$ , or  $\{a_1, a_5, a_{10}\}$ .

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose whether or not to include  $a_1$  into the subset - 2 ways (Yes or No)

Step 2. Choose whether or not to include  $a_2$  into the subset - 2 ways (Yes or No)

$\vdots$

Step  $|A|$ . Choose whether or not to include  $a_{|A|}$  into the subset - 2 ways (Yes or No)

By multiplication rule, we have that  $|\Omega| = 2 \times 2 \times \dots \times 2 = 2^{|A|}$ .

The choice of steps suggests that we could have selected a different format for our outcomes. Let us consider the following  $|A|$ -tuple: (In/Out for  $a_1$ , In/Out for  $a_2$ , ..., In/Out for  $a_{|A|}$ ). For example, a possible outcome would be: (In, Out, In, Out, ..., Out). This would correspond to the subset  $\{a_1, a_3\}$ .

We are always free to choose the format/definition of an outcome, as long as we correctly capture the information in the problem. This can be very helpful - choosing a different outcome format can open up a problem by being easier to construct.

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**Problem:** A typical PIN is a sequence of any four digit chosen from 10 numbers, with repetition allowed. How many different PINs are possible?

**Solution:** Let us consider what an outcome looks like for this problem. Let the outcome be the 4-tuple: (1st digit, 2nd digit, 3rd digit, 4th digit). An example outcome would be (1,2,3,4) which would correspond to the PIN 1234.

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose the digit for the first position - 10 ways

Step 2. Choose the digit for the second position - 10 ways

Step 3. Choose the digit for the third position - 10 ways

Step 4. Choose the digit for the fourth position - 10 ways

By the multiplication rule, we have that  $|\Omega| = 10^4$ .

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**Problem:** Suppose now that a PIN is allowed to be 4 digits or 6 digits. Now how many different PINs are possible?

**Solution:** Let us consider what an outcome looks like for this problem. We note that there are really 2 types of outcomes here. The first is the type of outcome that represents a 4 digit PIN: a 4-tuple like (1st digit, 2nd digit, 3rd digit, 4th digit). An example outcome would be (1,2,3,4) which would correspond to the PIN 1234. The second is the type of outcome that represents a 6 digit PIN: a 6-tuple like (1st digit, 2nd digit, 3rd digit, 4th digit, 5th digit, 6 digit). An example outcome would be (1,2,3,4,5,6) which would correspond to the PIN 123456.

We can consider these two types of outcomes separately, and then add the number of PINS of each type together.

We already have the result for the number of 4 digit PINs from above.

Let us think of how to construct a 6 digit outcome. We propose the following steps:

Step 1. Choose the digit for the first position - 10 ways

Step 2. Choose the digit for the second position - 10 ways

Step 3. Choose the digit for the third position - 10 ways

Step 4. Choose the digit for the fourth position - 10 ways

Step 5. Choose the digit for the fifth position - 10 ways

Step 6. Choose the digit for the sixth position - 10 ways

By the multiplication rule, we have that there are  $10^6$  distinct 6 digit pins.

Adding together, we have that  $|\Omega| = 10^4 + 10^6 = 1010000$ .

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The strategy that we used to solve the problem above can be generalized to something called the Addition Rule.

## Addition Rule

**Addition Rule for 2-Partitions:** Let  $A$  and  $B$  be sets, such that  $A \cap B = \emptyset$ . Note that  $\{A, B\}$  is a partition of the set  $A \cup B$ . The addition rule states that

$$|A \cup B| = |A| + |B|$$

We can further generalize this to any sized partition.

**Addition Rule for  $n$ -Partitions:** Let  $A$  be an arbitrary set, and  $\{A_1, A_2, \dots, A_n\}$  be an arbitrary partition of  $A$ . The addition rule states that

$$|A| = |A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$$

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Let us reframe the previous problem in a way that makes clear use of the addition rule.

- $\Omega$ : the set of all 4 and 6 digit PINs
- $A$ : the set of all 4 digit PINs
- $B$ : the set of all 6 digit PINs

Note that  $\Omega = A \cup B$  and  $A \cap B = \emptyset$ . Hence, by the addition rule, we have that  $|\Omega| = |A \cup B| = |A| + |B|$ .

We know that  $|A| = 10^4$  and  $|B| = 10^6$ . Hence,  $|\Omega| = 10^6 + 10^4$ .

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## Problem:

How many odd numbers between 1000 and 9999 have distinct digits?

## Solution:

Let us consider what an outcome looks like for this problem. Let the outcome be a 4-tuple (First digit, Second digit, Third digit, Fourth digit), where all digits are distinct, the fourth digit must be 1, 3, 5, 7, or 9, and the first digit cannot be 0. For example, one possible outcome would be (1, 2, 3, 7), which would correspond to the number 1237.

Attempt 1: Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose the first digit.

Step 2. Choose the second digit.

Step 3. Choose the third digit.

Step 4. Choose the fourth digit.

Observe that the number of ways of performing Step 4 depends upon the choices made in the earlier steps. For example, if the choices made in the first three steps are 1, 3, and 5, then Step 4 can be performed in two ways. However, if the choices made in the first three steps are 2, 4, and 6 then Step 4 can be performed in five ways. Hence, we cannot apply multiplication rule to solve the problem in the above manner.

Attempt 2: We propose the following steps:

Step 1. Choose the fourth digit.

Step 2. Choose the third digit.

Step 3. Choose the second digit.

Step 4. Choose the first digit.

Note that the number of ways of performing Step 4 depends upon whether a zero was chosen in the earlier steps. If a zero was chosen in either Step 2 or Step 3 then the number of ways of performing Step 4 is 7, otherwise it is 6. Hence, multiplication rule cannot be applied to solve the problem in the above manner.

Attempt 3: We propose the following steps:

Step 1. Choose the fourth digit.

Step 2. Choose the first digit.

Step 3. Choose the second digit.

Step 4. Choose the third digit.

There are 5 ways to perform Step 1, 8 ways to perform Step 2, 8 ways to perform Step 3, and 7 ways to perform Step 4. Note that the number of ways of doing each step is independent of the choices made in the earlier steps.

By the multiplication rule,  $|\Omega| = 5 \times 8 \times 8 \times 7 = 2240$ .

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**Problem:**

How many even numbers between 1000 and 9999 have distinct digits?

**Solution:**

Let us consider what an outcome looks like for this problem. Let the outcome be a 4-tuple (First digit, Second digit, Third digit, Fourth digit), where all digits are distinct, the fourth digit must be 2, 4, 6, 8, or 0, and the first digit cannot be 0. For example, one possible outcome would be (1, 2, 3, 0), which would correspond to the number 1230.

Attempt 1. Let us try the same steps as we used in the correct solution for counting the number of odd numbers.

Step 1. Choose the fourth digit.

Step 2. Choose the first digit.

Step 3. Choose the second digit.

Step 4. Choose the third digit.

Note that the number of ways to perform Step 2 depends on the choice in Step 1. If 0 is chosen in Step 1, then there are 9 choices for Step 2. However, if 2,4,6, or 8 is chosen in Step 1, then there are 8 choices for Step 2 (since neither 0 nor the selected digit in Step 1 can be used).

Attempt 2. Let us consider the following two sets:

- $A$ : the set of numbers that end in 0 from 1000 to 9999 with distinct digits
- $B$ : the set of numbers that end in 2,4,6,8 from 1000 to 9999 with distinct digits

Note that  $\Omega = A \cup B$  and  $A \cap B = \emptyset$ . Hence, we can use the addition rule to find  $|\Omega|$ .

Let us first determine  $|A|$ . Let us consider an outcome in this set. Let the outcome be a 4-tuple (First digit, Second digit, Third digit, Fourth digit), where all digits are distinct, the fourth digit must be 0, and the first digit cannot be 0. For example, one possible outcome would be (1,2,3,0), which would correspond to the number 1230.

We propose the following steps:

Step 1. Choose the fourth digit - 1 way (it must be 0)

Step 2. Choose the first digit - 9 ways (it can be any remaining number)

Step 3. Choose the second digit - 8 ways

Step 4. Choose the third digit - 7 ways

Note that now the number of ways to do each step is the same regardless of the choice made. By the multiplication rule, we have that  $|A| = 1 \times 9 \times 8 \times 7 = 504$ .

Let us now determine  $|B|$ . Let us consider an outcome in this set. Let the outcome be a 4-tuple (First digit, Second digit, Third digit, Fourth digit), where all digits are distinct, the fourth digit must be 2,4,6, or 8, and the first digit cannot be 0. For example, one possible outcome would be (1,2,3,8), which would correspond to the number 1238.

We propose the following steps:

Step 1. Choose the fourth digit - 4 ways (it must be 2,4,6, or 8)

Step 2. Choose the first digit - 8 ways (it cannot be the digit from Step 1 or 0)

Step 3. Choose the second digit - 8 ways

Step 4. Choose the third digit - 7 ways

Note that now the number of ways to do each step is the same regardless of the choice made. By the multiplication rule, we have that  $|B| = 4 \times 8 \times 8 \times 7 = 1792$ .

By the addition rule, we have that  $|\Omega| = |A| + |B| = 504 + 1792 = 2296$ .

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## Complementary Counting

Sometimes it can be difficult to count the number of outcomes with some sort of constraints, but easy to count the total number of outcomes without constraint and the total number of “bad” outcomes (outcomes that do not satisfy the constraint). In these cases, it is helpful to use a technique called complementary counting.

Let  $A$  be the set of outcomes that satisfy the constraints. Let  $B$  be the set of outcomes that do not satisfy the constraints. Note that  $A \cup B$  is the set of outcomes without constraint, and that  $A \cap B = \emptyset$ .

Then by the addition rule, we have that

$$|A| = |A \cup B| - |B|$$

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Let us see complementary counting in action by using it to solve the previous problem.

- $A$ : the set of outcomes that are even numbers from 1000 to 9999 with distinct digits
- $B$ : the set of outcomes that are odd numbers from 1000 to 9999 with distinct digits

Note that  $A \cup B$  is the set of all outcomes that are numbers from 1000 to 9999 with distinct digits.

Let us first determine  $|A \cup B|$ . Let us consider an outcome in this set. Let the outcome be a 4-tuple (First digit, Second digit, Third digit, Fourth digit), where all digits are distinct, and the first digit cannot be 0. For example, one possible outcome would be (1,2,3,0), which would correspond to the number 1230.

We propose the following steps:

- Step 1. Choose the first digit - 9 ways (it cannot be 0)
- Step 2. Choose the second digit - 9 ways (it can be any remaining number)
- Step 3. Choose the third digit - 8 ways
- Step 4. Choose the fourth digit - 7 ways

By the multiplication rule, we have that  $|A \cup B| = 9 \times 9 \times 8 \times 7 = 4536$ .

We know that  $|B| = 2240$  from before. Hence, we have that  $|A| = |A \cup B| - |B| = 4536 - 2240 = 2296$ .