

CMSC 250: Discrete Structures

Summer 2017

Lecture 9 - Outline

June 13, 2017

Permutations and Combinations

Permutations

A permutation of a set is an ordering of the elements of the set in a row. For example, the set $\{x, y, z\}$ has six permutations: $xyz, xzy, yxz, yzx, zxy, zyx$.

Problem:

Given an arbitrary set $A = \{a_1, a_2, \dots, a_n\}$, where $|A| = n$, how many permutations are there?

Solution: Let us consider what an outcome looks like for this problem. Let the outcome be the n -tuple: (1st element, 2nd element, \dots , n th element). An example of this would be: $(a_n, a_{n-1}, \dots, a_2, a_1)$.

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose a first element - n ways

Step 2. Choose a second element - $n - 1$ ways
(since we cannot choose the first element)

\vdots

Step n . Choose a n th element - 1 way (since we cannot choose any of the elements chosen in Steps 1 to $n - 1$)

By the multiplication rule, we have that $|\Omega| = n \times (n - 1) \times (n - 2) \times \cdots \times 1$.

This product is so often used and cumbersome to write down, that we normally denote it as $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 1$. It is read as n factorial.

Problem:

Consider the set of letters $\{a, b, c, d, e, f, g, h\}$.

- (a) How many possible permutations are there of these letters?

- (b) How many permutations of these letters contain the substring abc ?
- (c) How many permutations of these letters have the letters a, b, c next to them, but not necessarily in that order.

Solution.

- (a) There are 8 distinct elements and hence $8!$ permutations.
- (b) We consider the string abc as one unit and that along with the remaining elements amounts to 6 distinct elements. Hence there are $6!$ possible permutations.
- (c) Note that we can construct all of the permutations of interest by taking each permutation from (b) and permutating the positions of a, b, c . Note that for each permutation from (b) this can be done in $3!$ ways.

Hence there are $6! \times 3!$ such permutations.

Permutations of Selected Elements.

We looked at permutations of n elements out of the available n elements. Now we will consider permutations of r elements out of the available n elements.

Problem:

Let $P(n, r)$ denote the number of r -permutations of a set A , where $|A| = n$. What is the value of $P(n, r)$?

Solution:

Let us consider what an outcome looks like for this problem. Let the outcome be the r -tuple: (1st element, 2nd element, \dots , r th element). An example of this would be: $(a_r, a_{r-1}, \dots, a_2, a_1)$.

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose a first element - n ways

Step 2. Choose a second element - $n - 1$ ways
(since we cannot choose the first element)

\vdots

Step n . Choose a r th element - $n - r + 1$ way
(since we cannot choose any of the elements chosen in Steps 1 to $r - 1$)

By the multiplication rule, we have that $|\Omega| = P(n, r) = n \times (n - 1) \times (n - 2) \times \cdots \times (n - r + 2) \times (n - r + 1)$.

This product is useful but cumbersome, so it is often compacted into a neat fraction, shown below:

$$\begin{aligned} P(n, r) &= n \times n - 1 \times n - 2 \times \cdots \times n - r + 1 \\ &= \frac{n \times (n - 1) \times \cdots \times (n - r + 1) \times (n - r) \times \cdots \times 1}{n - r \times (n - r - 1) \times (n - r - 2) \times \cdots \times 1} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

Problem:

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different contestants?

Solution:

Let the set of contestants be the set $C = \{1, 2, \dots, 100\}$. Let us consider what an outcome looks like for this problem. Let the outcome be the 3-tuple: (1st place winner, 2nd place winner, 3rd place winner). An example of this would be: (3, 56, 54).

Notice that the outcomes are just 3-permutations of C . Hence, $|\Omega| = P(100, 3) = \frac{100!}{97!} = 100 \times 99 \times 98 = 970200$

Problem:

In how many ways can we order 26 letters of the alphabet so that no two of the vowels a, e, i, o, u occur next to each other?

Solution:

Let us consider what an outcome looks like for this problem. Let the outcome be the ordered pair where the left element is a 21-tuple and the right element is a 5-tuple: ((1st consonant, 2nd consonant, ..., 21st consonant), (Slot for a, Slot for e, Slot for i, Slot for o, Slot for u)). The slot number for each of the vowels indicates the position that the vowel should be inserted in the permu-

tation of consonants, and the slot numbers for each of the vowel must be distinct. The slot number of 1 indicates that the vowel should be inserted to the left of the 1st consonant, and the slot number of 22 indicates that the vowel should be inserted to the right of the last consonant. Note that this construction of an outcome enforces the constraint that vowels cannot be placed next to each other.

An example outcome would be $((b, c, d, f, \dots, x, y, z), (1, 4, 7, 12, 17))$ would correspond to the normal ordering of the alphabet.

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose a permutation of the consonants
- $21!$ ways

Step 2. Choose a 5-permutation of the set $\{1, 2, \dots, 22\}$
for the slots for the vowels - $P(22, 5)$
ways

By multiplication rule, we have that:

$$|\Omega| = 21! \times P(22, 5) = \frac{21! \times 22!}{17!}$$

Combinations

Let n and r be non-negative integers. An r -combination of a set of n elements means an unordered selection of r elements from the n elements of S . The symbol $\binom{n}{r}$ (read as “ n choose r ”) denotes the number of r -combinations of a set of n elements. This is same as the number of subsets of size r that can be chosen from a set of n elements.

Do you see the distinction between a r -permutation and a r -combination? A r -permutation is an *ordered* selection of r elements, i.e., both, which r elements, as well as the order in which they are chosen are important. Two r -permutations are the same if the r elements chosen are the same and they are chosen in the same order. In contrast, in a r -combination, only the choice of r elements is important. The order in which the r elements are chosen is irrelevant. Two r -combinations are the same if they have the same r elements regardless of the orders of selection of these elements.

Problem:

In general, what is the value of $\binom{n}{r}$, i.e., how many r -combinations are possible if we have a set of n distinct objects?

Solution:

We will answer this question by giving an expression that relates $\binom{n}{r}$ and $P(n, r)$.

A r -permutation can be obtained in two steps as follows.

Step 1. Choose r elements from the available n elements - $\binom{n}{r}$ ways

Step 2. Arrange the chosen r elements - $r!$

By the multiplication rule, the total number of r -permutation is given by

$$P(n, r) = \binom{n}{r} \times r!$$

Rearranging the terms of the above equation we get

$$\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Problem:

From a group of 8 men and 6 women, how many different teams can be formed consisting of 2 women and 2 men?

Solution: Let the set of women be $W = \{W_1, W_2, \dots, W_6\}$ and the set of men be $M = \{M_1, M_2, \dots, M_8\}$. Let us consider what an outcome looks like for this problem. Let the outcome be the ordered pair: (a 2-subset of W , a 2-subset of M).

For example, an outcome $(\{W_1, W_4\}, \{M_3, M_7\})$ would represent the team consisting of W_1 , W_4 , M_3 , and M_7 .

Let us think of how to construct an outcome for this problem. We propose the following steps:

Step 1. Choose a 2-subset of W - $\binom{6}{2}$ ways

Step 2. Choose a 2-subset of M - $\binom{8}{2}$ ways

Using the multiplication rule, the total number of possible teams is $\binom{6}{2} \times \binom{8}{2} = 420$.

Catalog of L^AT_EX Commands

$$\binom{n}{r} - \backslash\mathrm{binom}\{n\}\{r\}$$