Problem 1. Consider the sum

$$\sum_{k=1}^{n} k^3 .$$

(a) Use a non-integral method to show that the sum is between $n^4/20$ and n^4.

(b) Use the integral method to find better upper and lower bounds.

Problem 2. Consider an array of size eight with the numbers $30, 80, 50, 60, 20, 10, 70, 40$. Assume you execute quicksort using the version of partition from CLRS. [Note that an element can exchange with itself, which counts as one exchange.]

(a) What is the array after the first partition. How many comparisons did you use? How many exchanges?

(b) Show the left side after the next partition. How many comparisons did you use? How many exchanges?

(c) Show the right side after the next partition on that side. How many comparisons did you use? How many exchanges?

(d) What is the total number of comparisons in the entire algorithm? What is the total number of exchanges in the entire algorithm?

Problem 3. Assume you execute quicksort using the version of partition from CLRS. We are interested in the exact fewest number of comparisons quicksort will do as a function of n (i.e., its best case). Assume $n = 2^k - 1$ (where k is a natural number).

(a) (i) Give an example with $n = 3$ that does as few comparisons as possible.

(ii) Give an example with $n = 7$ that does as few comparisons as possible.

(iii) Give an example with $n = 15$ that does as few comparisons as possible.

(b) We might guess that the number of comparisons is approximately $n \lg(n+1) - 2n$. (Why?) Create a table with a column for $n = 1, 3, 7, 15$; a column with the exact number of comparisons for quicksort in the best case; a column with the value of the approximate guess, $n \lg(n + 1) - 2n$; and a column with the difference between the exact value and the approximate guess.

(c) Using the information from the above table give an exact formula for the number of comparisons.

(d) Write a recurrence for the number of comparisons in the best case as a function of n. (You may use your notes from class.)

(e) Use mathematical induction to prove that your formula is a solution to the recurrence.