Making Change for a $n$ Cents

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1 Introduction

How many ways are there to make change of a dollar using just pennies, nickels, dimes, and quarters? This is a well known question; however, the answers I found were disappointing. They were of two types:

1. There are 242 ways to make change. Here they are. (And then the author lists them.)

2. The number of ways to make $n$ cents change is the coefficient of $x^n$ in the Taylor series for

$$ \frac{1}{(1 - x)(1 - x^5)(1 - x^{10})(1 - x^{25})} $$

which can be worked out. (I have never seen it worked out.)

The first answer yields an actual number but is not interesting mathematically. The second answer is interesting mathematically but does not yield an actual number. In this paper I give a simple derivation of a closed form for the problem for $n$ cents. Along the way I obtain an actual number for the case of $n = 100$ in a simple way.

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2 Recurrences

Def 2.1

1. Let \( a_n \) be the number of ways to make change of \( n \) cents using pennies.

2. Let \( b_n \) be the number of ways to make change of \( n \) cents using pennies and nickels.

3. Let \( c_n \) be the number of ways to make change of \( n \) cents using pennies, nickels, and dimes.

4. Let \( d_n \) be the number of ways to make change of \( n \) cents using pennies, nickels, dimes, and quarters.

By convention \( a_0 = b_0 = c_0 = d_0 = 1 \).

We do one example: What is \( c_{15} \)?

1. If one dime is used then for the remaining five cents you must use either (1) one nickels, or (2) five pennies.

2. If zero dimes are used then for the remaining fifteen cents you must use either (1) three nickels, (2) two nickels and five pennies, (3) one nickels and ten pennies, or (4) zero nickels and fifteen pennies.

Hence \( c_{15} = 2 + 4 = 6 \).

How do \( a_n, b_n, c_n, d_n \) relate to each other? We explain one relation and then list the others which are similar.

Say you have \( n \geq 10 \) cents and you want to know \( c_n \). You will either use zero dimes or at least one dime. Hence \( c_n = b_n + c_{n-10} \). What if \( 5 \leq n \leq 9 \)? Then you cannot use dimes, hence \( c_n = b_n \). What if \( 0 \leq n \leq 4 \)? Then you cannot use a nickels or dimes, hence \( c_n = a_n \).

We now summarize the relations between \( a_n, b_n, c_n, d_n \). They are derived in a manner similar to the above discussion of \( c_n \).
1. For all \( n \), \( a_n = 1 \). This is obvious.

2. For \( 0 \leq n \leq 4 \), \( b_n = a_n = n \). For \( n \geq 5 \), \( b_n = a_n + b_{n-5} \).

3. For \( 0 \leq n \leq 4 \), \( c_n = a_n = n \). For \( 5 \leq n \leq 9 \), \( c_n = b_n \). For \( n \geq 10 \), \( c_n = b_n + c_{n-10} \).

4. For \( 0 \leq n \leq 4 \), \( d_n = a_n = n \). For \( 5 \leq n \leq 9 \), \( d_n = b_n \). For \( 10 \leq n \leq 24 \), \( d_n = c_n \). For \( n \geq 25 \), \( d_n = c_n + d_{n-25} \).

3 Closed Form for \( b_n \) And Making Change of a Dollar

Using the recurrence for \( b_n \) in Section 2, and \((\forall n)[a_n = 1]\), one can show

\[
(\forall n)[b_n = \left\lfloor \frac{n}{5} \right\rfloor + 1].
\]

Using the recurrences in Section 2 and use the formula for \( b_{5L} \) we can easily compute \( d_{100} \).

\[
d_{100} = c_{100} + c_{75} + c_{50} + c_{25} + c_0
\]

\[
c_0 = 1
\]

\[
c_{25} = b_{25} + b_{15} + b_5 = 6 + 4 + 2 = 12
\]

\[
c_{50} = b_{50} + b_{40} + b_{30} + b_{20} + b_{10} + b_0 = 11 + 9 + 7 + 5 + 3 + 1 = 36
\]

\[
c_{75} = b_{75} + b_{65} + b_{55} + b_{45} + b_{35} + c_{25} = 16 + 14 + 12 + 10 + 8 + 12 = 72
\]

\[
c_{100} = b_{100} + b_{90} + b_{80} + b_{70} + b_{60} + c_{50} = 21 + 19 + 17 + 15 + 13 + 36 = 121
\]

Hence

\[
d_{100} = 1 + 12 + 36 + 72 + 121 = 242.
\]

4 Closed Form for \( c_n \)

Let \( n = 5L + L_0 \) where \( 0 \leq L_0 \leq 4 \) and \( L \geq 1 \). Note that \( L = \left\lfloor \frac{n}{5} \right\rfloor \). It is easy to show that \( c_{5L+L_0} = c_{5L} \). Hence we have:
\[ c_n = c_{5L} = b_{5L} + c_{5L-10} = b_{5L} + b_{5(L-2)} + c_{5(L-4)} = b_{5(L-0)} + b_{5(L-2)} + \cdots + b_{5(L-2i)} + c_{5(L-2i-2)} = (L + 1) + (L - 1) + \cdots + (L - 2i + 1) + c_{5(L-2i-2)} \]

If \( L \) is even then let \( i = (L - 2)/2 \) to obtain

\[ c_n = (L + 1) + (L - 1) + \cdots + 3 + c_0 = (L + 1) + (L - 1) + \cdots + 3 + 1 = \frac{1}{4}(L^2 + 4L + 4). \]

If \( L \) is odd then let \( i = (L - 1)/2 \) to obtain

\[ c_n = (L + 1) + (L - 1) + \cdots + 4 + c_5 = (L + 1) + (L - 1) + \cdots + 4 + 2 = \frac{1}{4}(L^2 + 4L + 3). \]

We combine the \( L \) even and \( L \) odd cases to obtain

**Theorem 4.1** Let \( n = 5L + L_0 \) where \( 0 \leq L_0 \leq 4 \) and \( L \geq 1 \). (Note that \( L = \left\lfloor \frac{n}{5} \right\rfloor \).) Then

\[ c_n = \frac{1}{4} \left( L^2 + 4L + 3.5 + \frac{(-1)^L}{2} \right). \]

**Corollary 4.2** \( c_n = \frac{n^2}{100} + O(n) \).

**5 Closed Form for \( d_n \)**

Let \( n = 5(5L+M)+L_0 \) where \( 0 \leq L_0, M \leq 4 \) and \( L \geq 1 \). Note that \( L = \left\lfloor \frac{n}{25} \right\rfloor \), \( M = \left\lfloor \frac{n \mod 25}{5} \right\rfloor \), and \( L_0 = n \mod 5 \).

It is easy to show that \( d_{5(5L+5M)+L_0} = d_{5(5L+M)} \). Hence we have:
\[ d_n = d_5(5L+M) = c_5(5L+M) + d_5(5L+M-5 \times 1) \]
\[ = c_5(5L+M) + c_5(5L+M-5 \times 1) + d_5(5L+M-5 \times 2) \]
\[ = c_5(5L+M) + c_5(5L+M-5 \times 1) + c_5(5L+M-5 \times 2) + \cdots + c_5(M+5) + d_5M \]
\[ = c_5(5L+M) + c_5(5L+M-5 \times 1) + c_5(5L+M-5 \times 2) + \cdots + c_5(M+5) + c_5M \]
\[ = \sum_{i=0}^{L} c_5(M+5i) \]

Using the closed form for \( c_n \) from Theorem 4.1 we obtain

\[ \sum_{i=0}^{L} c_5(M+5i) = \frac{1}{4} \sum_{i=0}^{L} (M + 5i)^2 + 4(M + 5i) + 3.5 + \frac{(-1)^{M+5i}}{2} \]
\[ = \frac{1}{4} \left( (L + 1)(M^2 + 4M + 3.5) + 25 \sum_{i=0}^{L} i^2 + (10M + 20) \sum_{i=0}^{L} i + \frac{1}{2} \sum_{i=0}^{L} (-1)^{M+5i} \right) \]

Using the well known closed form for the summations \( \sum_{i=0}^{L} i^2 \) and \( \sum_{i=0}^{L} i \), and the easily seen closed form \( \sum_{i=0}^{L} (-1)^{M+5i} = \frac{(1+(-1)^L)(-1)^M}{2} \), we obtain

**Theorem 5.1** Let \( n = 5(5L + M) + L_0 \) where \( 0 \leq L_0, M \leq 4 \) and \( L \geq 1 \). (Note that \( L = \left\lfloor \frac{n}{25} \right\rfloor \), \( M = \left\lfloor \frac{n \text{ (mod 25)}}{5} \right\rfloor \), and \( L_0 = n \text{ (mod 5)} \).) Then

\[ d_n = \frac{(L + 1)(50L^2 + 30LM + 6M^2 + 85L + 24M + 21)}{24} + \frac{(1 + (-1)^L)(-1)^M}{16}. \]

**Corollary 5.2** \( d_n = \frac{n^3}{7500} + O(n^2) \)
6 Other Denominations

Let $S$ be a finite set of denominations of coins. Let $f_S(n)$ be the number of ways to make change of $n$ cents using coins with denominations in $S$. Does $f_S(n)$ always have a closed form? Using generating functions the answer is yes; however, this may be cumbersome to find. Does $f_S(n)$ always have a closed form that can be derived using recurrences (as we have done here for $S = \{1, 5, 10, 25\}$)? We believe yes.

It would be an interesting project to automate the process. That is, have a program that will, on input the set $S$ output the function $f_S(n)$ (e.g., output the formula for $d_n$ in Theorem 5.1).