A Short Proof for a Generalization of Vizing's Theorem

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ABSTRACT

For a simple graph of maximum degree Δ, it is always possible to color the edges with $\Delta + 1$ colors (Vizing); furthermore, if the set of vertices of maximum degree is independent, Δ colors suffice (Fournier). In this article, we give a short constructive proof of an extension of these results to multigraphs. Instead of considering several color interchanges along alternating chains (Vizing, Gupta), using counting arguments (Ehrenfeucht, Faber, Kierstead), or improving nonvalid colorings with Fournier's Lemma, the method of proof consists of using one single easy transformation, called "sequential recoloring", to augment a partial k-coloring of the edges.

Let G be a loopless multigraph of multiplicity $p \geq 1$ (that is: no more than p parallel edges can join a pair of vertices). The degree of a vertex x is denoted $d_G(x)$. Let $m_G(x, y)$ denote the number of parallel edges joining x and y; put

$$m_G(x) = \max_y m_G(x, y).$$

We shall prove the following result:

Theorem 1. Let G be a multigraph of maximum degree $\Delta \leq D$ and of multiplicity $p \leq t$. If the set

$$S = \{x/x \in V(G);\ d_G(x) = D;\ m_G(x) = t\}$$

is independent ("stable") or empty, then the edges of G can be colored with $D + t - 1$ colors.

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Clearly, with $D = \Delta + 1$, we obtain immediately the classical Vizing's theorem, and with $D = \Delta$, a theorem of Fournier (in fact, the main part of it, because in [3], it is possible to assume only that S induces an acyclic subgraph).

Proof. Assume that the result is valid for all graphs having less edges than G. If $S \neq \emptyset$, consider a vertex $x_0 \in S$ (otherwise, take for x_0 any vertex of maximum degree). Color with $D + t - 1$ colors all the edges of G but one edge $e_0 = [x_0, y_0]$ incident to x_0 (this is possible by the induction hypothesis). For a vertex y, let C_y be the set of colors of the edges incident to y. We shall define a sequence of distinct edges $e_0 = [x_0, y_0]$, $e_1 = [x_0, y_1], \dots, e_{k-1} = [x_0, y_{k-1}]$, all incident to x_0, together with a function f that associates to each edge e_i of the sequence a color $\alpha_{i+1} = f(e_i)$, according to the following iterative procedure.

(I) Let $\alpha_1 = f(e_0)$ be a color that $\notin C_{y_0}$; such a color does exist because $|C_{y_0}| < \Delta \leq D + t - 1$.

(II) If $\alpha_i \in C_{x_0}$ and if $\alpha_i \neq f(e_j)$ for all $j < i - 1$, consider the edge $e_i = [x_0, y_i]$ incident to x_0 that is colored with α_i; let $\alpha_{i+1} = f(e_i)$ be a color satisfying the two conditions:

(1) $\alpha_{i+1} \notin C_{y_i}$,

(2) $\alpha_{i+1} \neq f(e_j)$ for all $j < i$ with $y_j = y_i$.

Such a color α_{i+1} does exist, because the number q of colors excluded by (1) and (2) is at most $|C_{y_i}| + [m_G(x_0, y_i) - 1]$; since $y_i \notin S$ (because $x_0 \in S$), we have

$$q < |C_{y_i}| + m_G(x_0, y_i) \leq d_G(y_i) + m_G(x_0, y_i) \leq D + t - 1.$$

(III) If $\alpha_k \notin C_{x_0}$, or if $\alpha_k = f(e_j)$ for an index $j < k - 1$, we stop, and the sequence $e_0, e_1, \dots, e_{k-1}$ is achieved.

Clearly, because of (III), the e_i's are all distinct, and an achieved sequence will be obtained.

For every edge $e_i, 0 \leq i \leq k - 1$, and every color $\gamma \notin C_{x_0}$, a *sequential f-recoloring from* (e_i, γ) consists in changing the color of e_i to γ, the color of e_{i-1} to $f(e_{i-1})$, the color of e_{i-2} to $f(e_{i-2})$, etc., and by coloring the uncolored edge e_0 with $f(e_0) = \alpha_1$. Since the conditions (1) and (2) are satisfied, we see that if $\gamma \notin C_{x_0}$ and $\gamma \notin C_{y_i}$, this produces a valid edge coloring.

Case I. $\alpha_k \notin C_{x_0}$. Since $\alpha_k \notin C_{y_{k-1}}$ by (1) with $i = k - 1$, and since $\alpha_k \notin C_{x_0}$, the sequential f-recoloring from (e_{k-1}, α_k) produces a valid edge coloring of G with $D + t - 1$ colors.

Case II. $\alpha_k = \alpha_j$ for an index $j < k - 1$. Since α_k has been chosen according to the rule (2) with $i = k - 1$, we note that $y_{k-1} \neq y_{j-1}$.

Consider a color $\beta \notin C_{x_0}$, which does exist because

$$|C_{x_0}| < d_G(x_0) \leq \Delta \leq D + t - 1.$$

In the partial graph generated by the edges with colors β or α_k, the connected component containing y_{k-1} is a bicolor chain $\mu[y_{k-1}, z]$ having y_{k-1} as an endpoint (because $\alpha_k \notin C_{y_{k-1}}$ by (1)). The only possible edge incident to x_0 that could belong to $\mu[y_{k-1}, z]$ is the edge $e_j = [x_0, y_j]$ of color $\alpha_k = \alpha_j$; in this case, the other endpoint z of the chain is the vertex x_0 (because $\beta \notin C_{x_0}$) and *the chain* $\mu[y_{k-1}, z]$ *does not contain the vertex* y_{j-1} (otherwise y_{j-1} would be a third endpoint of this chain, a contradiction).

Consider a new edge coloring C' of $G - e_0$ obtained by interchanging the colors β and α_k in the bicolor chain $\mu[y_{k-1}, z]$.

If $z = x_0$, then $\alpha_k \notin C'_{x_0}$, $\alpha_k \notin C'_{y_{j-1}}$, and the sequential f-recoloring from (e_{j-1}, α_k) produces a valid edge coloring of G.

If $z = y_{j-1}$, then $\beta \notin C'_{y_{j-1}}$, $\beta \notin C'_{x_0}$, and the sequential f-recoloring from (e_{j-1}, β) produces a valid edge coloring of G.

If $z \neq x_0, z \neq y_{j-1}$, then $\beta \notin C'_{x_0}$, $\beta \notin C'_{y_{j-1}}$, and the sequential f-recoloring from (e_{k-1}, β) produces a valid edge coloring of G.

In each case, we get an edge coloring of G with $D + t - 1$ colors. QED.

Corollary 1 (Fournier [2]). Let G be a multigraph of maximum degree Δ and of multiplicity p. If the set of vertices of maximum degree is independent, then $\Delta + p - 1$ colors suffice to color the edge-set of G.

This follows from Theorem 1 with $D = \Delta$.

Corollary 2. Let G be a multigraph of maximum degree Δ and of multiplicity p, and let M be a maximal matching of G. The edges of G can be colored with $\Delta + p$ colors so that all the edges in M get the same color.

In the multigraph $G - M$ obtained from G by removing the edges of M, the vertices of degree Δ constitute an independent set (or an empty set), because of the maximality of M. By Corollary 1, the edges of $G - M$ can be colored with $\Delta + p - 1$ colors, and with one extra color for M, we obtain an edge coloring with $\Delta + p$ colors.

Remark. The proof of this theorem provides a fast (and simple) algorithm for the edge-coloring with only $\Delta + p$ colors. It suffices to use the first color for the edges of some maximal matching M, and to assign one of the remaining $\Delta + p - 1$ colors to each edge of $G - M$ as long as it does not violate the validity law of an edge-coloring. If an edge e_0 cannot be colored, we achieve the sequence $e_0, e_1, \ldots, e_{k-1}$, and one interchange followed by the sequential recoloring provides a better edge-coloring. If the number of edges of G is m, the number of elementary operations involved for $e_0 = [x, y]$ is at most $O(d_G(x)) + O(m)$. For all the edges incident to a vertex x, the number of operations is at most

$$d_G(x)\{O[d_G(x)] + O(m)\} = O[d_G(x)^2] + d_G(x)O(m).$$

For all the edges of G, the number of operations is at most

$$\sum O[d_G(x)^2] + \left\{\sum d_G(x)\right\}O(m) = O\left[\sum d_G(x)^2\right] + 2mO(m)$$

(the sum is over all the vertices x of G).

Since $\Sigma\, d_G(x)^2 \leq [\Sigma\, d_G(x)]^2 = 4m^2$, we have $O[\Sigma\, d_G(x)^2] = O(m^2)$; therefore the total number of operations is at most $O(m^2) + O(m^2) = O(m^2)$. Thus, the algorithm is polynomial (but it does not necessarily give an optimal coloring if the graph is of class 1).

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