# Edge coloring a multigraph

A multigraph is a graph in which multiple edges can exist between any two vertices.
The multiplicity M of a multigraph is the maximum number of edges joining two vertices.
The degree D of a multigraph is the maximum degree of a vertex in the graph.
Edge coloring means assigning a color to each edge so that no two adjacent edges have the same color
We would like to color the multigraph with as few colors as possible.

It is possible to color a multigraph with at most D+M colors.

Consider a multigraph in which some of the edges have been assigned colors and edge $(x,y_0)$ is not yet colored.
Color c is missing from vertex v if no edge adjacent to v is colored with color c.
Color c is present at vertex v if some edge adjacent to v is colored with color c.

If there is a color d that is missing from x and missing from $y_0$, then assign color d to edge $(x,y_0)$.
Otherwise, build a fan of k edges $(x,y_0)$, $(x,y_1)$,...,$(x,y_{k-1})$.
Each fan edge $(x,y_i)$ will have a potential future color $a_{i+1}$.
In this fan, it can happen that $y_i=y_j$ when $i\neq j$.
That is, two different edges $(x,y_i)$ and $(x,y_j)$ connect x to the same vertex $y_i=y_j=y$.
Let A(y) denote the set of potential future colors of the edges adjacent to vertex y.
For example, in the case that $y_i=y_j=y$, colors $a_{i+1}$ and $a_{j+1}$ are in A(y).

Build the fan according to the following rules:

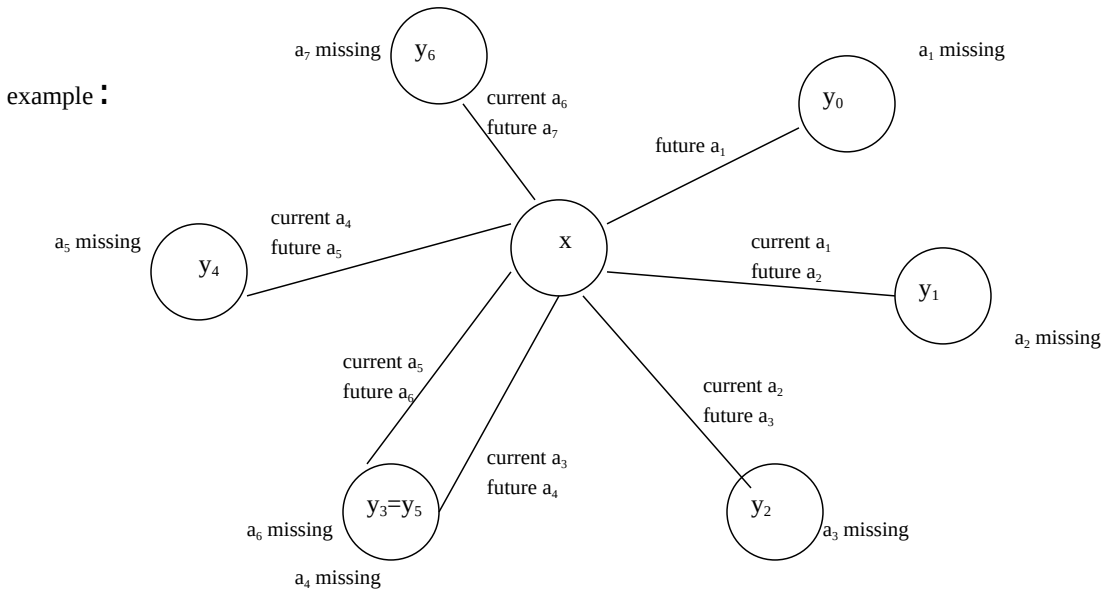
Let $a_1$ be a color missing from $y_0$.
Such a color exists because at most D-1 colors could be present at $y_0$.

For i>0,
If $a_i$ is missing from x, then stop building the fan,
or if $a_i$ is the same color as some color $a_j$ for some j<i, then stop building the fan.

If $a_i$ is present at x and $a_i$ is different from $a_1$, $a_2$,...,$a_{i-1}$ then keep building the fan.
$a_i$ is present at x, so we can pick the edge $(x,y_i)$ which is currently colored $a_i$.
Add this edge to the fan.
Now, find $a_{i+1}$, the potential future color of edge $(x,y_i)$.
$a_{i+1}$ should be a color which is missing from $y_i$ and which is not in the set $A(y_i)$.
Such a color exists because "missing from $y_i$" rules out at most D colors
and "not in $A(y_i)$" rules out at most M-1 colors.

Since the number of edges adjacent to x is finite, at some point the fan-building must stop.
We have a fan of k edges, and every edge $(x, y_j)$ has its current color $a_j$ and its potential future color $a_{j+1}$,
except for edge $(x,y_0)$, which has no current color, just the potential future color $a_1$.

example :

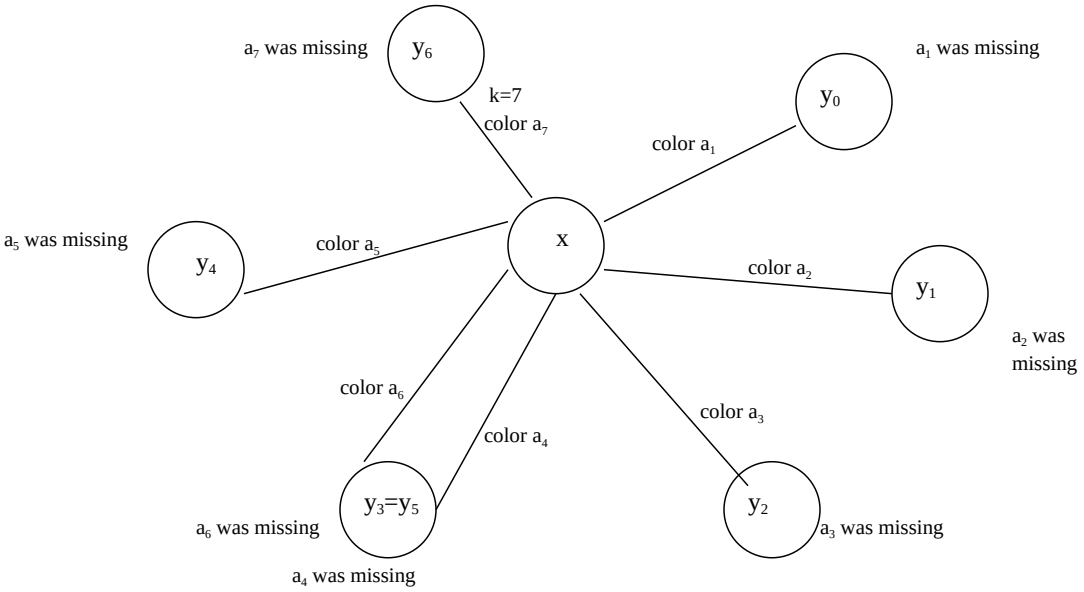


Now, if for some edge $(x,y_i)$ in the fan we had a color c missing from x and missing from $y_i$ then we could
color $(x,y_i)$ with color c
color $(x,y_{i-1})$ with color $a_i$
...
color $(x,y_0)$ with color $a_1$
to produce a valid edge coloring.

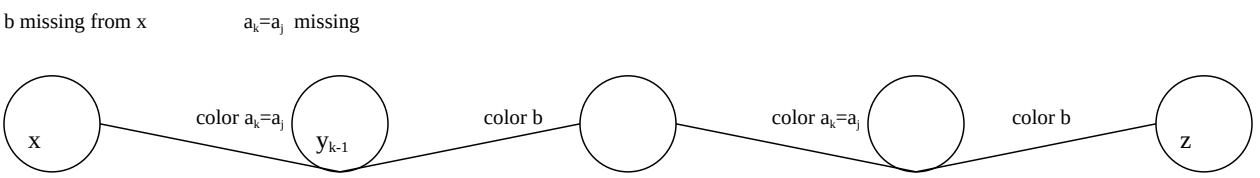
Next, we determine color c and the edge (x,y).

If we stopped the fan-building because $a_k$ is missing from x,
then $a_k$ is missing both from $y_{k-1}$ and from x, so
color $(x,y_{k-1})$ with $a_k$
color $(x,y_{k-2})$ with $a_{k-1}$
....
color $(x,y_0)$ with $a_1$.

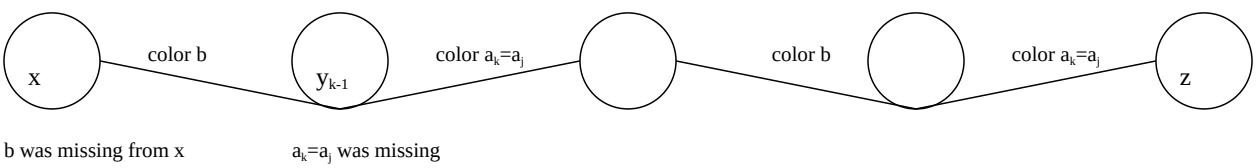
example: k=7, $a_7$ was missing from x




If we stopped fan-building because $a_k=a_j$ for some j<k, then let b be a color missing from x.
Follow a path of edges alternately colored b and $a_k$ from vertex $y_{k-1}$

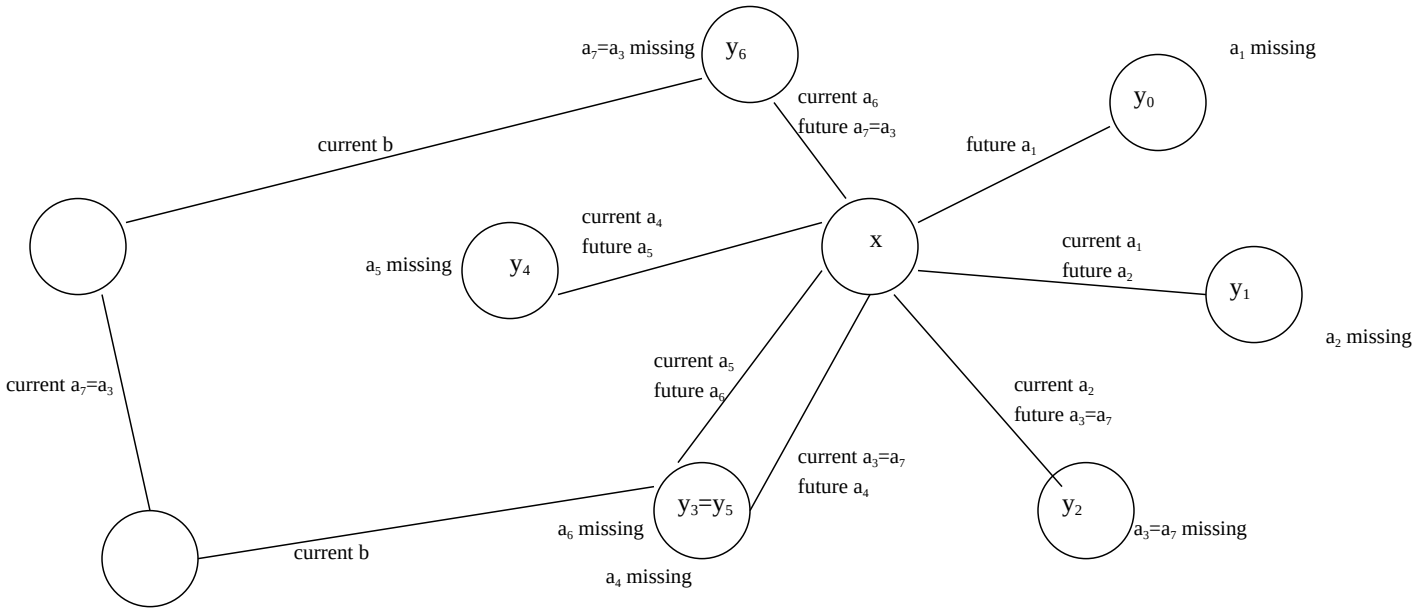


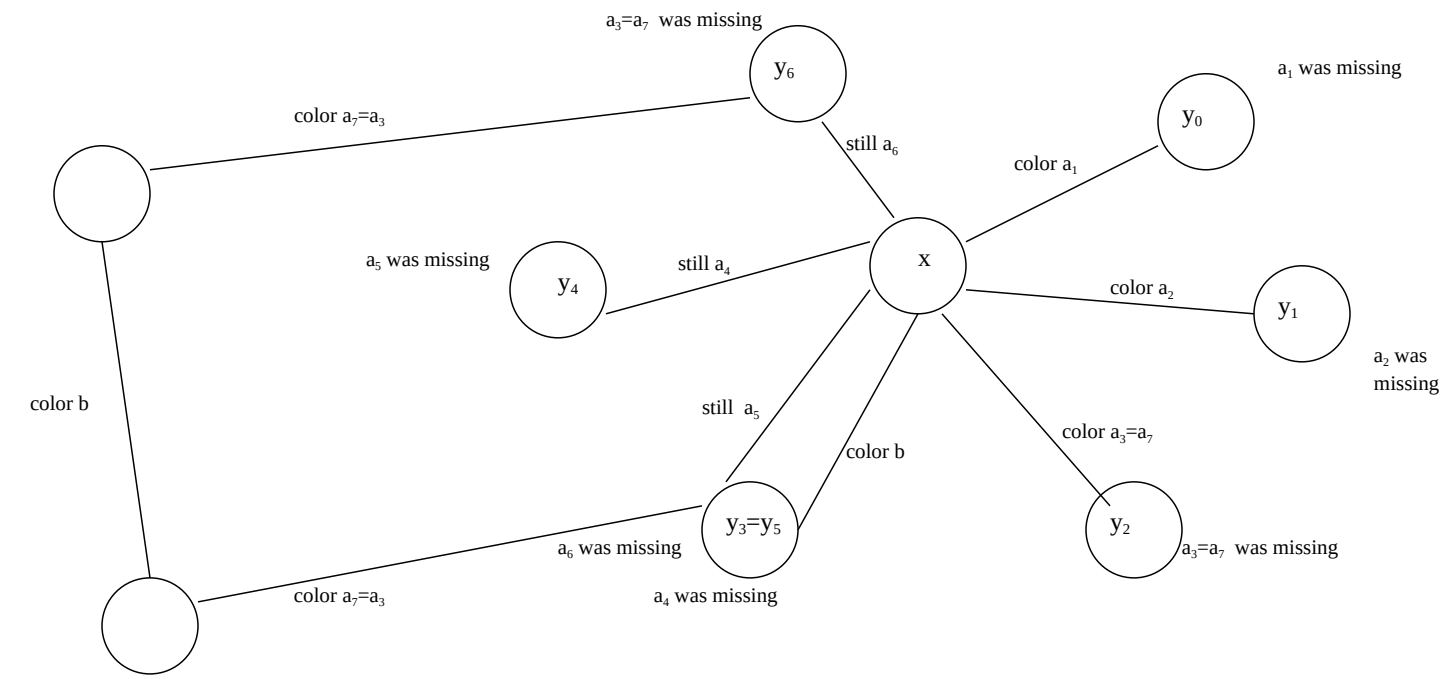

And then reverse the colors in this path.



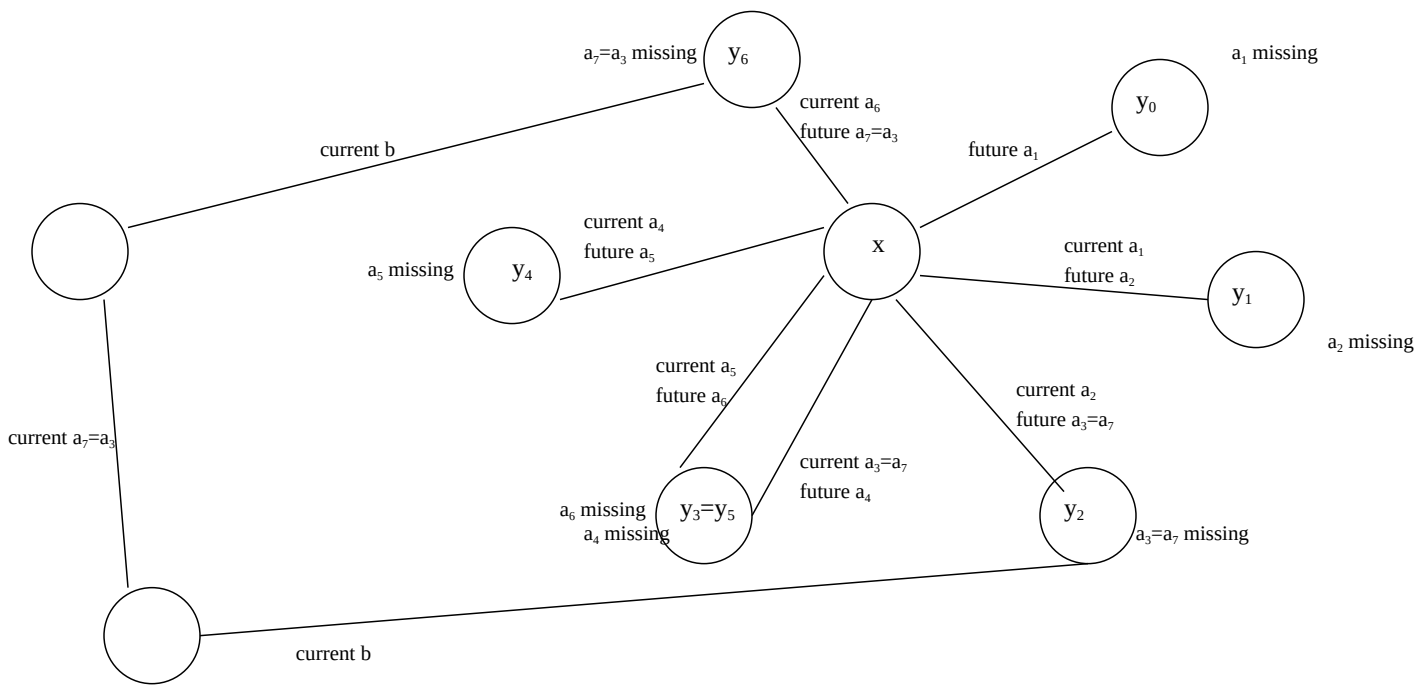

This path ends at some vertex z

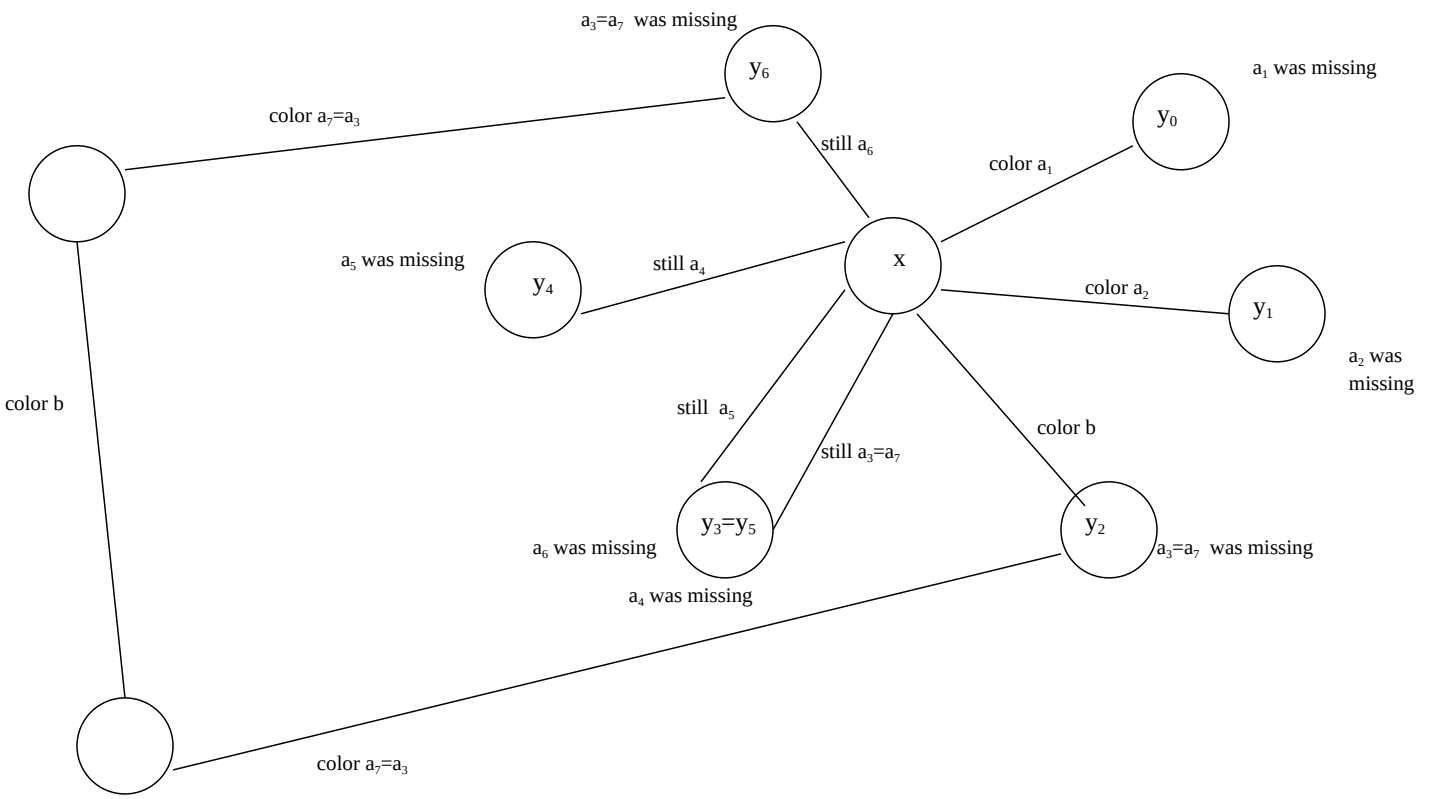
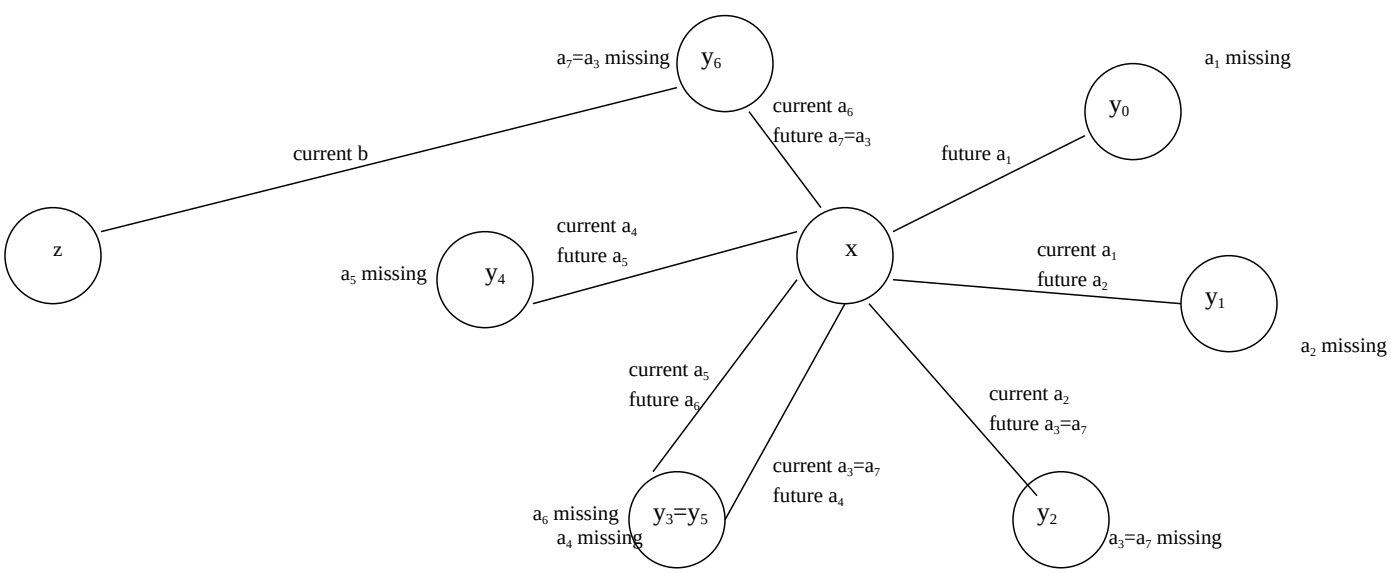
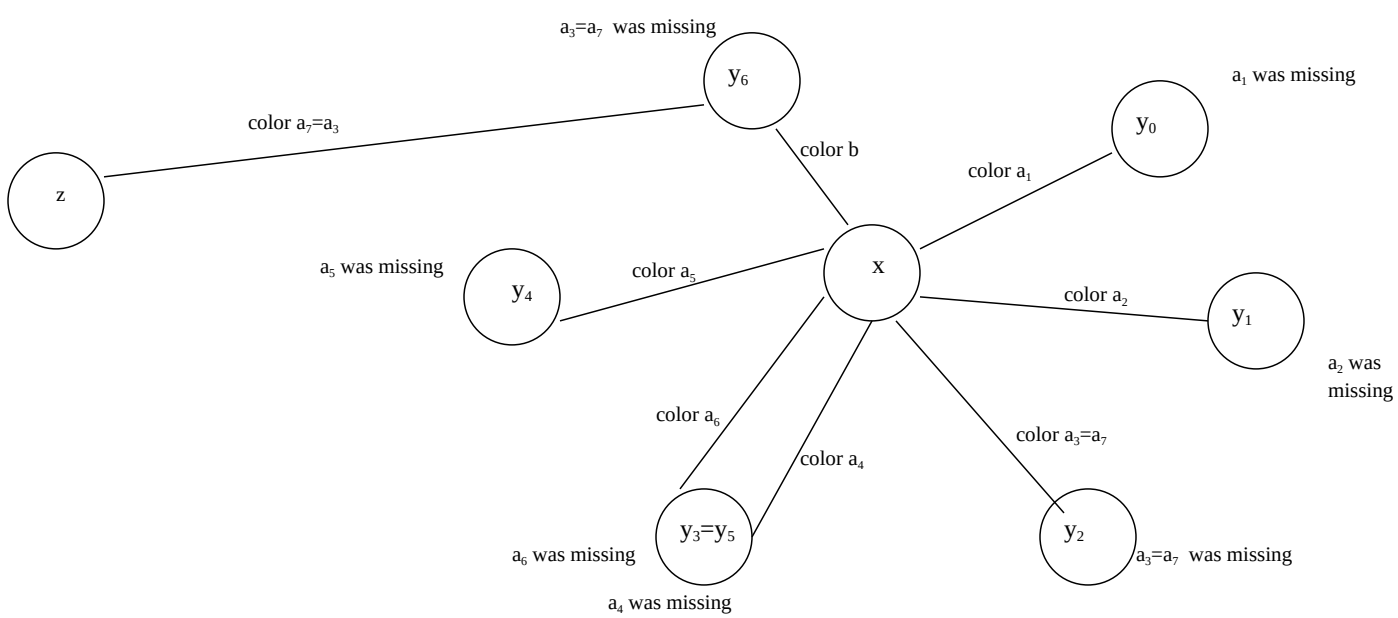
If z=x, that means the path reaches x with an edge of color $a_j$ so the path goes through vertex $y_j$.
When we reverse the colors in this path, we make b present at x, and we make $a_j$ missing from x.
Now $a_j$ is missing both from x and from $y_{j-1}$ so
color $(x,y_{j-1})$ with color $a_j$
color $(x,y_{j-2})$ with color $a_{j-1}$
...
color $(x,y_0)$ with color $a_1$



If $z=y_{j-1}$, that means the path reaches $y_{j-1}$ with an edge of color b.
When we reverse the colors in this path, we make $a_j$ present at $y_{j-1}$.
Now b is missing both from x and from $y_{j-1}$ so
color $(x,y_{j-1})$ with color b
color $(x,y_{j-2})$ with color $a_{j-1}$
...
color $(x,y_0)$ with color $a_1$






If $z \neq x$ and $z \neq y_{j-1}$ then.
when we reverse the colors in this path, we make b missing from $y_{k-1}$.
Now b is missing both from x and from $y_{k-1}$ so
color $(x,y_{k-1})$ with color b
color $(x,y_{k-2})$ with color $a_{k-1}$
...
color $(x,y_0)$ with color $a_1$