Advanced Numerical Linear Algebra Homework 2
AMSC/CMSC 763 Due October 12, 2017

1. We have seen in class that the GMRES algorithm constructs an approximation x_k to the solution of a linear system $Ax = b$ where x_k has the form

$$x_k = x_0 + V_k y_k \, ,$$

y_k is the solution of a least squares problem

$$\min_y \left\| \|r_0\|_2 e_1 - \widehat{H}_k y \right\|_2$$

e_1 is a unit vector of size $k+1$, and $\widehat{H}_k$ is an upper-Hessenberg matrix with $k+1$ rows and k columns. Show that the residual $r_k = b - Ax_k$ satisfies $\|r_k\|_2 = h_{k+1,k}[y_k]_k$, which means that once x_k is constructed, this residual norm is available at essentially no cost.
Hint: Derive a QR-factorization of $\widehat{H}_k$ that can be used to solve the least squares problem and that shows the structure of the residual norm.

Problem 2. Let $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ be generated by the Arnoldi process for a matrix A, with square Hessenberg matrix H_k.

a. Suppose $v_{k+1} = 0$. Show that in this case, the eigenvalues of H_k are eigenvalues of A, and identify the associated eigenvectors.

b. More generally, if $v_{k+1} \neq 0$, μ is an eigenvalue of H_k and an *estimate* for an eigenvector of A is generated in a manner anagous to what is done for part (a), show that the residual $Av - \mu v$ is orthogonal to the Krylov subspace $K_k(A, v_1)$.

c. What happens to H_k when A is symmetric?

Problem 3. Suppose $A\mathbf{u} = \mathbf{f}$ is a linear system of equations in which the coefficient matrix A is symmetric and positive-definite. Let $Q = GG^T$ be a symmetric positive-definite preconditioner; note that no assumption is made about the structure of G other than that Q admits a factorization of this type. Given this (formal) factorization, we could apply the unpreconditioned conjugate gradient algorithm to

$$G^{-1}AG^{-T}\mathbf{v} = G^{-1}\mathbf{f}, \quad \mathbf{v} = G^T\mathbf{u}.$$

Use this fact to derive the *preconditioned conjugate gradient algorithm* (PCG) given below. This shows that the extra computation required by PCG at each step is a solution of a system with coefficient matrix Q. This may or may not depend on the factorization.

THE PRECONDITIONED CONJUGATE GRADIENT METHOD

Choose $\mathbf{u}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - A\mathbf{u}^{(0)}$, solve $Q\mathbf{z}^{(0)} = \mathbf{r}^{(0)}$, set $\mathbf{p}^{(0)} = \mathbf{z}^{(0)}$

`for` $k = 0$ `until` convergence `do`

  $\alpha_k = \langle \mathbf{z}^{(k)}, \mathbf{r}^{(k)} \rangle / \langle A\mathbf{p}^{(k)}, \mathbf{p}^{(k)} \rangle$

  $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \alpha_k \mathbf{p}^{(k)}$

  $\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - \alpha_k A\mathbf{p}^{(k)}$

  <Test for convergence>

  Solve $Q\mathbf{z}^{(k+1)} = \mathbf{r}^{(k+1)}$

  $\beta_k = \langle \mathbf{z}^{(k+1)}, \mathbf{r}^{(k+1)} \rangle / \langle \mathbf{z}^{(k)}, \mathbf{r}^{(k)} \rangle$

  $\mathbf{p}^{(k+1)} = \mathbf{z}^{(k+1)} + \beta_k \mathbf{p}^{(k)}$

`enddo`

4. A demo given in class showed the effect damped Jacobi smoothing had on the discrete one-dimensional diffusion equation.

a. Implement this demo yourself. That is, show that a few steps of damped Jacobi smoothing makes the error smooth. You can generate the matrix and right-hand side using the code

```
e1 = ones(n,1);
h = 1/(n+1);
A = spdiags([-e1 2*e1 -e1], [-1,0,1], n, n)/h;
f = h*e1;
```

Reasonable choices for n are 31 or 63, but feel free to play with anything you like. To make the case, start with a random initial value and then plot the error in one or two figures.

b. Continue this experiment by implementing the *two-grid* algorithm. This will require construction of the coarse-grid matrix A_{2h} and the prolongation and restriction operators, P and R. You can then take one step of the two-grid algorithm to consist of two smoothing steps, followed by restriction, coarse-grid correction, and prolongation. Show that this algorithm displays "textbook" multigrid behavior.