Advanced Numerical Linear Algebra Homework 5
AMSC/CMSC 763 Due December 7, 2017

Problem 1.

a. If the eigenvalues of a symmetric indefinite matrix $\mathcal{A}$ lie in two intervals $[-a, -b\,] \cup [c, d\,]$, with positive constants a, b, c, and d, then the residuals generated by the unpreconditioned MINRES method satisfy

$$\|r^{(2k)}\|_2 \leq 2\left(\frac{\sqrt{ad}-\sqrt{bc}}{\sqrt{ad}+\sqrt{bc}}\right)\|r^{(0)}\|_2.$$

Suppose that $a = d$ and $b = c$ so that the inclusion intervals are symmetrically placed on either side of the origin. Show how the bound above simplifies in this case.

b. Show that the eigenvalues of the symmetric positive-definite matrix $\mathcal{A}^T\mathcal{A} = \mathcal{A}^2$ lie in the interval $[c^2, d^2]$. Compare your bound from part (a) to the convergence bound for the conjugate gradient method appled to $\mathcal{A}^2 x = \mathcal{A}b$.

2. Let $A_h\mathbf{u} = \mathbf{f}$ be a system of equations defined on a geometric grid of width h, and let A_{2h} denote the analogous matrix defined on a coarse grid of width $2h$. A_{2h} could come directly from a discretization, or it could be defined as RA_hP where P is a grid-transfer operator mapping vectors on the coarse grid to those on the fine grid, and $R = P^T$. The two-grid algorithm with smoothing operator M is:

Algorithm 2.5: TWO-GRID ITERATION
Choose $\mathbf{u}^{(0)}$
`for` $i = 0$ `until` convergence
  `for` k `steps` $\mathbf{u}^{(i)} \leftarrow (I - M^{-1}A)\,\mathbf{u}^{(i)} + M^{-1}\mathbf{f}$ (presmoothing)
  $\overline{\mathbf{r}} = R(\mathbf{f} - A\mathbf{u}^{(i)})$ (restrict residual)
  $A_{2h}\,\overline{\mathbf{c}} = \overline{\mathbf{r}}$ (solve for coarse grid correction)
  $\mathbf{u}^{(i)} \leftarrow \mathbf{u}^{(i)} + P\overline{\mathbf{c}}$ (prolong and update)
  $\mathbf{u}^{(i)} \leftarrow (I - M^{-1}A)\,\mathbf{u}^{(i)} + M^{-1}\mathbf{f}$ (postsmoothing)
  $\mathbf{u}^{(i+1)} \leftarrow \mathbf{u}^{(i)}$ (update for next iteration)
`end`

Let $\mathbf{e}^{(i)} = \mathbf{u} - \mathbf{u}^{(i)}$. Show that

$$\begin{aligned}\mathbf{e}^{(i+1)} &= (I - M^{-1}A)^k(A^{-1} - PA_{2h}^{-1}R)A(I - M^{-1}A)^k\mathbf{e}^{(i)} \\ &= (I - M^{-1}A)^k(A^{-1} - PA_{2h}^{-1}R)(I - AM^{-1})^kA\mathbf{e}^{(i)}.\end{aligned} \tag{1}$$

For this, you will need to show that $A(I - M^{-1}A)^k = (I - AM^{-1})^kA$.
Assuming M is symmetric, use the second expression of (1) to show that $\mathbf{e}^{(i+1)} = (I - M_{MG}^{-1}A)\mathbf{e}^{(i)}$ where M_{MG} is symmetric. This means that one step of the two-grid computation can be viewed as a preconditioning operator M_{MG}.