The Algebraic Degree of $\cos\left(\frac{a\pi}{m}\right)$

1 Introduction

In high school we are taught the following:

1. $\cos(\pi/1) = -1$
2. $\cos(\pi/2) = 0$
3. $\cos(\pi/3) = \frac{1}{2}$
4. $\cos(\pi/4) = \frac{\sqrt{2}}{2}$.
5. $\cos(\pi/6) = \frac{\sqrt{3}}{2}$.

Note that $\cos(\pi/5)$ is missing. In Harold Boas's paper [1] he shows that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$$

which is twice the golden ratio. That paper is the inspiration for this paper.

Are numbers of the form $\cos(a\pi/b)$ with $a, b \in \mathsf{N}$ always algebraic? Yes. This is well known. In this paper we prove the theorem with an eye towards (a) getting explicit polynomials, and (b) seeing what the degree of those polynomials are.

2 Chebyshev Polynomials of the First Kind

Def 2.1 The Chebyshev Polynomials of the first kind is, for all $n \in \mathsf{N}$,

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k} x.$$

The following is well known.

Theorem 2.2 *For all $n \in \mathsf{N}$, $T_n(\cos(\theta)) = \cos(n\theta)$.*

3 When is $\cos\left(\frac{a\pi}{b}\right) = \cos\left(\frac{na\pi}{b}\right)$?

Lemma 3.1

1. *Let $n \in \mathsf{N}$. For all*

$$\theta \in \left\{\frac{2k\pi}{n-1} : k \in \mathsf{Z}\right\} \bigcup \left\{\frac{2k\pi}{n+1} : k \in \mathsf{Z}\right\}$$

$\cos(\theta) = \cos(n\theta)$. *(If $n = 1$ then just take the second unionand.)*

2. (a) *If n is odd then the n roots of $T_n(x) - x = 0$ are*

$$\left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-1}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n-1}{2}\right\}.$$

(If $n = 1$ then just take the second unionand.)

(b) *If n is even then the n roots of $T_n(x) - x = 0$ are*

$$\left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-2}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n}{2}\right\}.$$

Proof:
1)

For the first unionand notice that

$$\cos\left(\frac{2k\pi}{n-1}\right) = \cos\left(\frac{2k\pi}{n-1} + 2k\pi\right) = \cos\left(\frac{n2k\pi}{n-1}\right).$$

For the second unionand notice that

$$\cos\left(\frac{2k\pi}{n+1}\right) = \cos\left(-\frac{2k\pi}{n+1}\right) = \cos\left(2\pi k - \frac{2k\pi}{n+1}\right) = \cos\left(\frac{n2k\pi}{n+1}\right).$$

2a) By Theorem 2.2 and Part 1 we have that all of the elements in

$$X = \left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \le k \le \frac{n-1}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \le k \le \frac{n-1}{2}\right\}$$

are roots of $T_n(x) - x = 0$. By algebra one can see that all of the angles mentioned are distinct and in $[0, \pi]$. Since cosine is injective on $[0, \pi]$, X has n different numbers Since $T_n(x) - x$ is of degree n, the elements of X are its n roots.

2b) Similar to the proof of 2a. ∎

4 A Polynomial for $\cos(\frac{k\pi}{m})$: m Odd

Def 4.1 Let $m \in \mathsf{N}$, m odd. Then

$$f_{\text{odd}}(m) = \left|\left\{k\colon \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k,m) = 1\right\}\right|.$$

Theorem 4.2 *Let $m \in \mathsf{N}$, $m \ge 3$, m odd.*

1. *There exists $p_m(x) \in \mathsf{Z}[x]$ of degree $f_{\text{odd}}(m)$ whose roots are*

$$A_{m,1} = \left\{\cos\left(\frac{2k\pi}{m}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k,m) = 1\right\}$$

2. *There exists $q_m(x) \in \mathsf{Z}[x]$ of degree $f_{\text{odd}}(m)$ whose roots are*

$$A_{m,2} = \left\{\cos\left(\frac{(m-2k)\pi}{m}\right) : \left(1 \le k \le \frac{m-1}{2}\right) \wedge \gcd(k,m) = 1\right\}$$

3. *All of the numbers in*

$$\left\{\cos\left(\frac{k\pi}{m}\right) : (1 \le k \le m-1) \wedge \gcd(k,m) = 1\right\}$$

are algebraic of degree $\le f_{\text{odd}}(m)$. (This follows from Parts 1 and 2.)

Proof:
1) We prove this by induction on m.
Base Case: $m = 3$. Then $A_{m,1} = \{\cos(2\pi/3)\} = \{-1/2\}$ and $A_{m,2} = \{\cos(\pi/3)\} = \{1/2\}$. Let $p_3(x) = 2x + 1$ and $q_3(x) = -2x + 1$. Both $p_3(x)$ and $q_3(x)$ are of degree $f_{\text{odd}}(3) = 1$.
Inductive Hypothesis The theorem is true for all odd m', $3 \le m' < m$.
Inductive Step Since m is odd, $m - 1$ is even. By Lemma 3.1.2 the $m - 1$ roots of $T_{m-1}(x) - x = 0$ are

$$\left\{\cos\left(\frac{2k\pi}{m-2}\right) : 0 \le k \le \frac{m-3}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{m}\right) : 1 \le k \le \frac{m-1}{2}\right\}.$$

We partition this set into the following disjoint sets based on the denominators after reducing fractions.

- The $k = 0$ case yields $\cos(0) = 1$. Hence $x - 1$ divides $T_{m-1}(x) - x$.

- For all $3 \leq m' \leq m-2$ such that m' divides $m-2$ we have

$$A_{m',1} = \left\{ \cos\left(\frac{2k\pi}{m'}\right) : \left(1 \leq k \leq \frac{m'-1}{2}\right) \wedge \gcd(k, m') = 1 \right\}.$$

  Since m is odd and m' divides $m-2$, m' is odd. By the induction hypothesis there is a polynomial $p_{m'}(x) \in \mathsf{Z}[x]$ of degree $f_{\text{odd}}(m')$ whose roots are the elements of $A_{m',1}$. Hence $p_{m'}(x)$ divides $T_{m-1}(x) - x$. (We will not be using that $p_{m'}(x)$ has degree $f_{\text{odd}}(m')$.)

- For all $3 \leq m' \leq m-1$ such that m' divides m we have

$$A_{m',1} = \left\{ \cos\left(\frac{2k\pi}{m'}\right) : \left(1 \leq k \leq \frac{m'-1}{2}\right) \wedge \gcd(k, m') = 1 \right\}.$$

  Since m is odd and m' divides m, m' is odd. By the induction hypothesis there is a polynomial $p_{m'}(x) \in \mathsf{Z}[x]$ of degree $f_{\text{odd}}(m')$ whose oots are the elements of $A_{m',1}$. Hence $p_{m'}$ divides $T_{m-1}(x) - x$. (We will not be using that $p_{m'}(x)$ has degree $f_{\text{odd}}(m')$.)

- $$\left\{ \cos\left(\frac{2k\pi}{m}\right) : \left(1 \leq k \leq \frac{m-1}{2}\right) \wedge \gcd(k, m) = 1 \right\}$$

  Note that this is the set $A_{m,1}$.

The polynomials

$$\{p_{m'}(x)\}_{(3 \leq m' \leq m-2) \wedge (m' | m-2)} \bigcup \{p_{m'}(x)\}_{(3 \leq m' \leq m-1) \wedge (m' | m)}$$

have disjoint sets of roots. Hence they are all relatively prime to each other. Clearly they all divide $T_{m-1}(x) - x$. Hence

$$p_m(x) = \frac{T_{m-1}(x) - x}{(x-1)(\prod_{3 \leq m' \leq m-2, m' | m-2} p_{m'}(x))(\prod_{3 \leq m' \leq m-1, m' | m} p_{m'}(x))}$$

is in $\mathsf{Z}[x]$ and its roots are the set $A_{m,1}$. Clearly the degree of $p_m(x)$ is $f_{\text{odd}}(m)$.

2) Let k be such that $1 \leq k \leq \frac{m-1}{2}$ and $\gcd(k, m) = 1$. Note that

$$\cos\left(\frac{(m-2k)\pi}{m}\right) = \cos\left(\frac{-(m-2k)\pi}{m}\right) = -\cos\left(\pi - \frac{-(m-2k)\pi}{m}\right) = -\cos\left(\frac{2k\pi}{m}\right)$$

Hence $A_{m,2} = -A_{m,1}$. Therefore we can take $q_m(x) = p_m(-x)$. ▮

Corollary 4.3 *Let $m \in \mathsf{N}$, $m \geq 3$, be odd. There exists a polynomial $r_m(x) \in \mathsf{Z}[x]$ of degree $m-1$ whose roots are*

$$\left\{\cos\left(\frac{k\pi}{m}\right) : \left(1 \leq k \leq m-1\right)\right\}.$$

Proof:

$$r_m(x) = \prod_{m' \geq 3, m' | m} p_{m'}(x).$$

∎

5 A Polynomial for $\cos\left(\frac{k\pi}{m}\right)$: m Even

Def 5.1 Let $m \geq 1$. Then

$$f_{\text{even}}(m) = |\{k : (1 \leq k \leq m \wedge \gcd(k,m) = 1\}|.$$

(We will only use this definition when m is even.)

Theorem 5.2 *Let $m \in \mathsf{N}$, $m \geq 2$, m even. There exists $q_m(x) \in \mathsf{Z}[x]$ of degree $f_{\text{even}}(m)$ whose roots are*

$$B_{m,1} = \left\{\cos\left(\frac{k\pi}{m}\right) : \left(1 \leq k \leq m\right) \wedge \gcd(k,m) = 1\right\}$$

Proof:
1) We prove this by induction on m.
Base Case: $m = 2$. Then $B_{m,1} = \{\cos(\pi/2\} = \{0\}$. Let $s_2(x) = x$. $s_2(x)$ has degree 1.
Inductive Hypothesis The theorem is true for all even m', $2 \leq m' < m$.
Inductive Step Since m is even, $m-1$ is even. By Lemma 3.1.2 the $2m-1$ roots of $T_{2m-1}(x) - x = 0$ are

$$\left\{\cos\left(\frac{2k\pi}{2m-2}\right) : 1 \leq k \leq \frac{m-1}{2}\right\} \bigcup \left\{\cos\left(\frac{2k\pi}{2m}\right) : 0 \leq k \leq \frac{m-3}{2}\right\}$$

which is

$$\left\{\cos\left(\frac{k\pi}{m-1}\right) : 1 \leq k \leq \frac{m-1}{2}\right\} \bigcup \left\{\cos\left(\frac{k\pi}{m}\right) : 0 \leq k \leq \frac{m-3}{2}\right\}$$

Since m is even, $m-1$ is odd. Therefore, by Corollary 4.3, there exists $r_m(x) \in \mathsf{Z}[x]$ whose roots are the elements of the first unionand.

We partition the second unionand into the following disjoint sets based on the denominators after reducing fractions.

- The $k=0$ case. This is just $\cos(0)=1$. Hence $x-1$ divides $T_{2m-1}(x)-x$.
- For all $2 \le m' \le m-1$ such that m' divides m we have

$$C_{m'} = \left\{ \cos\left(\frac{k\pi}{m'}\right) : \left(1 \le k \le m'-1\right) \wedge \gcd(k,m')=1 \right\}.$$

  There are two cases.

  1. If m' is odd then, by Theorem 4.2.3, there exists $r_{m'}(x) \in \mathsf{Z}[x]$ whose roots are $C_{m'}$. Clearly $r_{m'}(x)$ divides $T_{2m-1}(x)-1$.
  2. If m' is even then, by the induction hypothesis, there exists $s_{m'}(x) \in \mathsf{Z}[x]$ whose roots are $C_{m'}$. Clearly $s_{m'}(x)$ divides $T_{2m-1}(x)-1$.

  Let $t_{m'}(x)$ be $r_{m'}(x)$ if m' is odd, and $s_{m'}(x)$ if m' is even.

- $$\left\{ \cos\left(\frac{k\pi}{m}\right) : \left(1 \le k \le m\right) \wedge \gcd(k,m)=1 \right\}$$

  This is $B_{m,1}$.

The polynomials

$$\{t_{m'}(x)\}_{(3 \le m' \le m-2) \wedge (m' | m-2)}$$

have disjoint sets of roots which are also disjoint from the roots of $x-1$. Hence all these polynomials are relatively prime to each other. Clearly they all divide $T_{m-1}(x)-x$. Hence

$$s_m(x) = \frac{T_{m-1}(x)-x}{(x-1)\prod_{2 \le m' \le m-1, m'|m-1} s_{m'}(x)}$$

Clearly the degree of this polynomial is $f_{\text{even}}(m)$. ∎

A The First 11 Chebyshev Polynomials

1. $T_2(x) = 2x^2 - 1$
2. $T_3(x) = 4x^3 - 3x$
3. $T_4(x) = 8x^4 - 8x^2$
4. $T_5(x) = 16x^5 - 20x^3 + 5x$
5. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
6. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

7. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
8. $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
9. $T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$
10. $T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 - 220x^3 - 11x$
11. $T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

B Table of Polynomials

In the first column if we have a number line $\pi/4$ we mean $\cos(\pi/4)$.

Roots	Poly
$\pi/2$	$x-1$
$\pi/3$ $2\pi/3$	$-2x+1$ $2x+1$
$\pi/4, 3\pi/4$	$2x^2-1$
$\pi/5, 3\pi/5$ $2\pi/5, 4\pi/5$	$4x^2-2x-1$ $4x^2+2x-1$
$\pi/6, 5\pi/6$	
$\pi/7, 3\pi/7, 5\pi/7$ $2\pi/7, 4\pi/7, 6\pi/7$	$-8x^3+4x^2+4x-1$ $8x^3+4x^2-4x-1$
$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$	
$\pi/9, 5\pi/9, 7\pi/9,$ $2\pi/9, 4\pi/9, 8\pi/9$	$-8x^3-6x+1$ $8x^3-6x+1$
$\pi/10, 3\pi/10, 7\pi/10, 9\pi/10$	
$\pi/11, 3\pi/11, 5\pi/11, 7\pi/11, 9\pi/11$ $2\pi/11, 4\pi/11, 6\pi/11, 8\pi/11, 10\pi/11$	$-32x^4+16x^4+32x^3-12x^2-6x+1$ $32x^5+16x^4-32x^3-12x^2+6x+1$

C Using $\cos(2\theta)$ to get $\cos(a\pi/3)$

By Lemma 3.1.2 the 2 roots of $T_2(x) - x = 0$ are

$$\{\cos(0)\} \cup \{\cos(2\pi/3)\}.$$

Since $T_2(x) - x = 2x^2 - x - 1 = (x-1)(2x+1)$ we have

$$\{1, -1/2\} = \{\cos(0), \cos(2\pi/3)\}.$$

Since $\cos(0) = 1$, we have

1. $\cos(2\pi/3) = -\frac{1}{2}$. Root of $2x+1$.
2. $\cos(\pi/3) = -\cos(\pi - \pi/3) = -\cos(2\pi/3) = \frac{1}{2}$. Root of $-2x+1$.

D Using $\cos(3\theta)$ to get $\cos(\pi/2)$

By Lemma 3.1.2 the 3 roots of $T_3(x) - x = 0$ are

$$\{\cos(0), \cos(\pi)\} \cup \{\cos(\pi/2)\}.$$

Since $T_3(x) - x = 4x^2 - 4x = 4x(x-1)(x+1)$

$$\{0, 1, -1\} = \{\cos(0), \cos(\pi/2), \cos(\pi)\}.$$

We know $\cos(0) = 1$ and $\cos(\pi) = -1$, so

1. $\cos(\pi/2) = 1$. Root of $x - 1$.

E Using $\cos(4\theta)$ to get $\cos(a\pi/5)$

By Lemma 3.1.2 the 4 roots of $T_4(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/3)\} \cup \{\cos(2\pi/5), \cos(4\pi/5)\}.$$

Since

$$(1) \quad T_4(x) - x = 8x^4 - 8x^2 - x + 1 = (2x+1)(x-1)(4x^2 + 2x - 1),$$

(2) $\cos(0) = 1$, and (3) $\cos(2\pi/3) = -1/2$, we have that $\cos(2\pi/5)$ and $\cos(4\pi/5)$ are roots of $6x^2 + 2x - 1$. Since $\cos(2\pi/5) > 0$ and $\cos(4\pi/5) < 0$ we have $\cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$ and $\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}$. With this we have:

1. $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$. Root of $4x^2 - 2x - 1$.
2. $\cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$. Root of $4x^2 + 2x - 1$.
3. $\cos(3\pi/5) = \frac{1-\sqrt{5}}{4}$. Root of $4x^2 - 2x - 1$.
4. $\cos(4\pi/5) = \frac{-1-\sqrt{5}}{4}$. Root of $4x^2 + 2x - 1$.

F Using $\cos(5\theta)$ to get not much

By Lemma 3.1.2 the 5 roots of $T_5(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/2), \cos(\pi)\} \cup \{\cos(\pi/3), \cos(2\pi/3)\}.$$

This will not give us any cosines we don't already know. Darn!
Even so, we note that

$$T_5(x) - x = 16x^5 - 20x^3 + 4x = 4x(x-1)x(x+1)(2x-1)(2x+1)$$

G Using $\cos(6\theta)$ to get $\cos(a\pi/7)$

By Lemma 3.1.2 the 6 roots of $T_6(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/5), \cos(4\pi/5)\} \cup \{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\}.$$

Since

(1) $T_6(x) - x = 32x^6 - 48x^4 + 18x^2 - x - 1 = (4x^2 + 2x - 1)(x - 1)(8x^3 + 4x^2 - 4x - 1)$,

(2) $\{\cos(2\pi/5), \cos(4\pi/5)\}$ are roots of $6x^2 + 2x - 1$, and (3) $\cos(0)$ is a root of $x - 1 = 0$, we have that

$$\{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\} \text{ are the 3 roots of } 8x^3 + 4x^2 - 4x - 1.$$

1. $\cos(\pi/7)$, $\cos(3\pi/7)$, $\cos(5\pi/7)$ are roots of $-8x^3 + 4x^2 + 4x - 1$.
2. $\cos(2\pi/7)$, $\cos(4\pi/7)$, $\cos(6\pi/7)$ are roots of $8x^3 + 4x^2 - 4x - 1$.

H Using $\cos(7\theta)$ to get $\cos(a\pi/4)$

By Lemma 3.1.2 the 7 roots of $T_7(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/3), \cos(2\pi/3), \cos(\pi)\} \cup \{\cos(\pi/4), \cos(\pi/2), \cos(3\pi/4)\}.$$

Since

(1) $T_7(x) - x = 64x^7 - 112x^5 + 56x^3 - 8x = 8(x - 1)(x + 1)(2x - 1)(2x + 1)x(2x^2 - 1)$.

(2) $\cos(0)$, $\cos(\pi)$, $\cos(\pi/3)$, $\cos(2\pi/3)$, $\cos(\pi/2)$ are, respectively, roots of $x-1$, $x+1$, $2x-1$, $2x + 1$, x, we have that

$$\{\cos(\pi/4), \cos(3\pi/4)\} \text{ are the 2 roots of } 2x^2 - 1.$$

Since $\cos(\pi/4) > 0$ and $\cos(3\pi/4) < 0$ we have the following:

1. $\cos(\pi/4) = \sqrt{2}/2$. Root of $2x^2 - 1$.
2. $\cos(2\pi/4) = 0$ is a root of x.
3. $\cos(3\pi/4) = -\sqrt{2}/2$ is a root of $2x^2 - 1$.
4. $\cos(4\pi/4) = -1$ is a root of $x + 1 = 0$.

I Using $\cos(8\theta)$ to get $\cos(a\pi/9)$

By Lemma 3.1.2 the 8 roots of $T_8(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\} \cup \{\cos(2\pi/9), \cos(4\pi/9), \cos(2\pi/3), \cos(8\pi/9\}.$$

Since

(1) $T_8(x)-x = 128x^8-256x^6+160x^4-32x^2-x+1 = (x-1)(8x^3+4x^2-4x-1)(8x^3-6x+1)(2x+1)$

(2) $\{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\}$ are roots of $8x^3 + 4x^2 - 4x - 1$, (3) $\cos(0)$ is a root of $x - 1 = 0$, and (4) $\cos(2\pi/3)$ is a root of $2x + 1$ we have

$$\{\cos(2\pi/9), \cos(4\pi/9), \cos(8\pi/9)\} \text{ are the 3 roots of } 8x^3 - 6x + 1.$$

1. $\cos(\pi/9)$, $\cos(5\pi/7)$, $\cos(7\pi/9)$ are roots of $-8x^3 + 6x + 1$.
2. $\cos(2\pi/9)$, $\cos(4\pi/9)$, $\cos(8\pi/9)$ are roots of $8x^3 - 6x + 1$.
3. $\cos(3\pi/9)$ is a root of $2x - 1$.
4. $\cos(6\pi/9)$ is a root of $2x + 1$.

J Using $\cos(9\theta)$ to Get Nothing New

By Lemma 3.1.2 the 9 roots of $T_9(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/4), \cos(\pi/2), \cos(3\pi/4), \cos(\pi)\} \cup \{\cos(\pi/5), \cos(2\pi/5), \cos(3\pi/5), \cos(4\pi/5\}.$$

This will not give us any cosines we don't already know. Darn! Even so, we note that Since

$$T_9(x)-x = 256x^9-576x^7+432x^5-120x^3+8x = (x-1)(2x^2-1)8x(x+1)(4x^2-2x-1)(4x^2+2x-1).$$

K Using $\cos(10\theta)$ to get $\cos(a\pi/11)$

By Lemma 3.1.2 the 10 roots of $T_{10}(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/9), \cos(4\pi/9), \cos(2\pi/3), \cos(8\pi/9)\}\cup$$

$$\{\cos(2\pi/11), \cos(4\pi/11), \cos(6\pi/11), \cos(8\pi/11), \cos(10\pi/11)\}.$$

Since

$$(1) \quad T_{10}(x)-x = 512x^{10}-1280x^{8}+1120x^{6}-400x^{4}+50x^{2}-1 = (x-1)(2x+1)(8x^{3}-6x+1)(32x^{5}+16x^{4}-32x^{3}-$$

(2) $\cos(0)$ is a root of $x-1$, (3) $\cos(2\pi/3)$ is a root of $2x+1$, (4) $\cos(2\pi/9)$, $\cos(4\pi/9)$, $\cos(8\pi/9)$ are the roots of $8x^3-6x+1$, we have that

$$\{\cos(2\pi/11), \cos(4\pi/11), \cos(6\pi/11), \cos(8\pi/11), \cos(10\pi/11)\}$$

are the 5 roots of $32x^5+16x^4-32x^3-12x^2+6x+1$.

L For $a, b \in \mathsf{N}$, $\cos(a\pi/b)$ is Algebraic

The above examples show that all of the number $\cos(\pi/2)$, $\cos(\pi/3)$, $\cos(\pi/4)$, $\cos(\pi/5)$, and $\cos(\pi/6)$ are algebraic. Are all numbers of the form $\cos(\pi/b)$ algebraic? If so then easily so are all numbers of the form $\cos(a\pi/b)$. The answer is Yes.

Theorem L.1

1. *For all $n \in \mathsf{N}$, $\cos(\pi/n)$ is algebraic.*
2. *For all $n, m \in \mathsf{N}$, $\cos(m\pi/n)$ is algebraic. (This follows from part 1 and the cosine addition formula.*
3. *For all $n, m \in \mathsf{N}$, $\sin(m\pi/n)$ is algebraic. (This follows from part 2 and the relationship between sine and cosine, and the closure of the algebraic numbers under square roots.)*
4. *For all $n, m \in \mathsf{N}$, $\tan(m\pi/n)$ is algebraic. (This follows from parts 2,3 and the relationship between tangent, sine, cosine, and the closure of the algebraic numbers under quotients.)*

Proof:

Let $n \in \mathsf{N}$.
As noted above,

$$T_n(\cos(n\theta) = \cos(\theta).$$

Let $\theta = \frac{\pi}{n}$. Then you get

$$T_n(\cos(\frac{\pi}{n}) = \cos(\pi) = -1.$$

On the right hand side you get a poly in $\cos(\frac{\pi}{n})$ with integer coefficients. Hence $\cos(\frac{\pi}{n})$ is algebraic. ∎

M What about Sin?

The following is due to jflipp on

Theorem M.1 $\sin(2\theta)$ *can not be written as a polynomial over* R *in* $\sin(\theta)$*.*

Proof: Assume, by way of contradiction, that there exists a polynomial $p(x) \in \mathsf{R}[x]$ such that $\sin(2\theta) = p(\sin(\theta))$. Since $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ we have

$$2\cos(\theta)\sin(\theta) = \sin(2\theta) = p(\sin(\theta)).$$

Note that if $\theta = 0$ then the left hand side is 0, so $p(\sin(0)) = 0$. Hence $p(0) = 0$. Therefore there exists $q(x) \in \mathsf{R}[x]$ such that $p(x) = xq(x)$. So

$$2\cos(\theta)\sin(\theta) = p(\sin(\theta)) = \sin(\theta)q(\sin(\theta)).$$

$$2\cos(\theta) = p(\sin(\theta)) = q(\sin(\theta)).$$

(We divided by $\sin(\theta)$ so we needed to have $\theta \notin \{n\pi\colon n \in \mathsf{Z}\}$; however, by continuity the two expressions are equal for all θ.)

Square both sides and use $\cos^2(\theta) = 1 - \sin^2(\theta)$ to get

$$4(1 - \sin^2(\theta)) = q(\sin(\theta))^2.$$

The two polynomials $4(1 - x^2)$ and $q(x)^2$ agree for infinitely many x, namely $\sin(\theta)$ as $\theta \in [0, \pi)$. Hence they are equal. But $q(x)^2$ is a square of a polynomial, and $4(1 - x^2) = 4(1 - x)(1 + x)$ is not. Contradiction. ∎

References

[1] H. Boas. The oldest trig in the book. *College Mathematics Journal*, 50(1):9–20, 2019.