Homework 6 MORALLY Due Mar 28 at 9:00AM
WARNING: THIS HW IS SIX PAGES LONG!!!!!!!!!!!!!!!!!

1. (0 points but please DO IT) What is your name?

2. (40 points) Given x, y, z $a_n(x, y, z)$ be defined by

   $a_1 = x$

   $a_2 = y$

   $a_3 = z$

   $(\forall n \geq 4)[a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}]$.

   (a) (20 points) Find α (AN APPROXIMATION TO 5 PLACES! THE CLOSED FORM IS UGLY!) such that
   $(\forall n \geq 1)[a_n \sim \alpha^n]$
   This α should only depend on the recurrence and not on x, y, z.
   (You will get three values of α. Two of them will be complex numbers of length < 1 so as n goes to infinity they are negligible. The third will be a real > 1 and that is what we want.)

   (b) (0 points but you will need this) Write a program that will, given x, y, z, n, generate
   $$a_1, a_2, \ldots, a_n$$

   (c) (0 points but you will need this) Write a program that will, given n, generate
   $$\alpha^1, \alpha^2, \ldots, \alpha^n.$$

   (d) (0 points but you will need this) Write a program that will, given x, y, z, n, run the two programs above and then generate
   $$\frac{\alpha^1}{a_1}, \frac{\alpha^2}{a_2}, \ldots, \frac{\alpha^n}{a_n}.$$
   This sequence should be roughly constant. Call that constant $C(x, y, z)$.

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(e) (20 points) For $1 \leq x, y, z \leq 5$ find $C(x, y, z)$. Put it into a table like this:

x	y	z	$C(x, y, z)$
1	1	1	
1	1	2	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	5	5	

(You will not have the $\vdots$ and you will have the $C(x, y, z)$ column filled in.

(f) (0 points) Which of x, y, z affects $C(x, y, z)$ the most? In what direction?

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3. (40 points) Given n we want to write n as a sum of cubes of INTEGERS. Note that

   - 23 is the sum of 9 cubes of NATURALS, but NOT 8.
   - 23 is the sum of 5 cubes of INTEGERS: $23 = 3^3+(-1)^3+4\times(-1)^3$.

   Let $n \in \mathsf{N}$. NCUZ(n) is the least number such that n can be written as the sum of NCUZ(n) cubes of INTEGERS. Clearly NCUZ(n) $\le n$.

   In this problem you will write a programs that will, given $n \in \mathsf{N}$, find $A[n]$, a bound on NCUZ(n).

   Note the following:

   n can be written as the sum of k cubes IFF $-n$ can be written as the sum of k cubes.

   Consider the following thought experiment:

   You want to find $A[23]$. So you look at using

   1^3: So then the answer would be $1 + A[23 - 1^3] = 1 + A[22]$.

   2^3: So then the answer would be $1 + A[23 - 2^3] = 1 + A[15]$.

   3^3: So then the answer would be $1 + A[23 - 3^3] = 1 + A[-4] = 1 + A[4]$.

   4^3: So then the answer would be $1+A[23-4^3] = 1+A[-41] = 1+A[41]$.

   CAN'T USE THIS- we do not know $A[41]$ while doing $A[23]$.

   We can use j^3 so long as $|23 - j^3| \le 22$.

   More generally, while looking at $A[i]$ can use j such that $|i-j^3| \le i-1$.

   On the next page we ask the question we want formally.

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(a) (10 points) Show that if $-n$ is the sum of k integer cubes, then n is the sum of k integer cubes.

(b) (0 points but you need to do this for the next part.) Write a program that will, given n, find, for all $0 \le i \le n$, a number $A[i]$ such that i can be written as the sum of $A[i]$ cubes of INTEGERS.

Here is how the program will work.

- $A[0] \leftarrow 0$ (0 can be written as the sum of 0 cubes).
- $A[1] \leftarrow 1$ (1 can be written as the sum of 1 cube).
- For $i \leftarrow 2$ to n

$$A[i] = 1 + \min\{A[|i - j^3|] \colon 0 \le |i - j^3| \le i - 1\}.$$

(c) (10 points) Run the program on $n = 10000$, so you get $A[1]$,. . .,$A[10000]$.

i. How many numbers took 1 cube? (For how many $1 \le i \le 10000$ is $A[i] = 1$?)

ii. How many numbers took 2 cubes? (For how many $1 \le i \le 10000$ is $A[i] = 2$?)

iii. How many numbers took 3 cubes? (For how many $1 \le i \le 10000$ is $A[i] = 3$?)

iv. ETC- until no numbers required that many cubes.

(d) (20 points) Email your code to Emily. She will run it on many numbers so make sure it is correct

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4. (20 points) On April Fools Day Bill wants to pull the following trick on Emily:

   Bill gives Emily a sequence $1, 2, 3$, asks her to guess the next number.

   She will (quite reasonably) guess 4.

   Bill will then produce a polynomial p such that

   $p(1) = 1$

   $p(2) = 2$

   $p(3) = 3$

   $p(4) = 1000$

   and then say FOOLED YOU! The answer was 1000. (Emily will then roll her eyes.)

   Help Bill out! Give a polynomial p with those values. Try to make the degree as low as possible.

   You CAN use tools on the web. If so, tell us what they are.

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5. (Extra Credit)

(a) Prove for all n, for all $a \geq 2^n$, DUP wins $(L_{2^n}, \mathsf{N} + \mathsf{Z} + \mathsf{N}^*; n)$

(b) Prove for all n, for all $a \geq 2^n$, DUP wins $(L_{2^n}, \mathsf{N} + \mathsf{Z} + \mathsf{Z} + \mathsf{N}^*; n)$