**Untimed Midterm. Morally Due Feb 28, 9:00AM**
**WARNING: THIS MIDTERM IS FOUR PAGES LONG!!!!!!!!!!!!!!!!!**

1. (10 points) In this problem $L_n$ is the linear order on $n$ points. In this problem you may assume that,

   *for all $k$ there is a $n$ such that DUP wins the DUP-SPOILER game with $L = L_n$ and $L' = \mathsf{N} + \mathsf{N}^*$ and $k$ moves. (In class we agreed that $n$ was roughly $2^k$. For this problem it does not matter what $n$ is.)* (Recall that $\mathsf{N}^*$ is $\mathsf{N}$ backwards $\ldots, 4, 3, 2, 1, 0$.

   (a) Give a strategy for Duplicator to win the DUP-SPOILER game with

   $L = \mathsf{N}$

   $L' = \mathsf{N} + \mathsf{Z}$

   Any value of $k$

   (b) Give a strategy for Duplicator to win the DUP-SPOILER game with

   $L = \mathsf{Z}$

   $L' = \mathsf{Z} + \mathsf{Z}$

   Any value of $k$.

   (You may use the last part in this part.)

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2. (20 points) In this problem you will write a program that, given $n, m$, outputs a formula on $n$ variables that has *exactly* $m$ satisfying assignments. That is an informal description which I formalize soon.

   You will be outputting boolean formulas in DNF form. We illustrate the format we want with an examples.

   If your formula is

$$(x_1 \land x_2 \land \neg x_3) \lor (\neg x_1 \land x_{10})$$

   you output

$$(1, 2, n3)(n1, 10).$$

   The *length* of a formula is the sum of the following (and we give examples from the above formula).

   The number of parenthesis: 4.

   The number of commas: 3.

   The number of $n$'s: 2.

   The number of occurences of numbers: 5. This is 1,2,3,1,10. Note that we don't count the 10 as two characters.

   The length of the above formula is $4 + 3 + 2 + 5 = 14$.

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(a) (0 points but you need it for the next part) Write a program that will, on input $n, m$ output one of the following:

- If $m = 0$ then output the formula $x \wedge \neg x$. (Note that this has 0 satisfying assignments.)
- If $m \geq 2^n + 1$ then output THERE IS NO SUCH FORMULA. (Note that there really is no such formula.)
- If $1 \leq m \leq 2^n$ then output a DNF formula on $n$ variables that has exactly $m$ satisfiying assignments. (There may be many, just output one of them.) Also output the formulas length.

(b) (5 points) Run your program on (2,2), (3,3), (4,4), ... (10,10) and plot the graph of $n$ vs. the length of what you get on input $(n, n)$. Hand in your data and the graph.

(c) (0 points) Speculate what the function $n$ goes to length of the formula for $(n, n)$ roughly is.

(d) (15 points) Email Emily your code. She will run it on many values so make sure that it is correct.

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3. (20 points) Let $n \in \mathsf{N}$. NCU($n$) is the least number such that $n$ can be written as the sum of NCU($n$) cubes. Clearly NCU($n$) $\le n$ since

$$n = 1^3 + \cdots + 1^3.$$

It is known that NCU($n$) $\le 9$.

In this problem you will write one programs that will, given $n \in \mathsf{N}$, find a bound on NCU($n$).

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(a) (0 points but you need to do this for the next part.) Write a program that will, given $n$, find, for all $1 \le i \le n$, a number $A[i]$ such that $i$ can be written as the sum of $A[i]$ cubes.

- $A[0] \leftarrow 0$ (0 can be written as the sum of 0 cubes).
- $A[1] \leftarrow 1$ (1 can be written as the sum of 1 cube).
- For $i \leftarrow 2$ to $n$

$$A[i] \leftarrow 1 + \min\{A[i - j^3] \colon i - j^3 \ge 0\}$$

  (We are speculating that if we used $j^3$ there is $i - j^3$ left.)

(b) (4 points- 1 points each) Run the program on $n = 1, 2, \ldots, 1000$.

  i. How many required 9 cubes? List them all. (23 should be one of them.)

  ii. How many required 8 cubes? List all that are $\le 100$. (50 should be one of them.)

  iii. Write 50 as the sum of 8 cubes.

  iv. Prove that 50 cannot be written as the sum of 7 cubes. (You may use that 23 cannot be written as the sum of 8 cubes.)

(c) (16 points) Email your code to Emily. She will run it on many numbers so make sure it is correct