Midterm One, March 9 8:00PM-10:00PM
WARNING: THIS MID IS SIX PAGES LONG!!!!!!!!!!!!!!!!!

1. (a) Let p and q be distinct primes. Let $n = p^2q^3$. Show that, $n^{2/5} \notin \mathsf{Q}$. USE Unique Factorization.

SOLUTION

1) Assume, BWOC, that $n^{2/5} = \frac{a}{b}$. So

$$n^2 = \frac{a^5}{b^5}.$$

$$n^2b^5 = a^5.$$

$$p^2q^3b^5 = a^5.$$

Let $p_1, \ldots, p_L$ be all of the primes that divide either a or b. (We do not know or care if p or q is one of the p_i's.) Then by Unique Factorization there is a unique $a_1, \ldots, a_L$ and $b_1, \ldots, b_L$ such that

$$a = p_1^{a_1} \cdots p_L^{a_L}$$

$$b = p_1^{b_1} \cdots p_L^{b_L}$$

So

$$p^2q^3p_1^{5b_1} \cdots p_L^{5b_L} = p_1^{5a_1} \cdots p_L^{5a_L}.$$

Let $LHSp$ be the number of times p appears on the LHS. Clearly $LHSp \equiv 2 \pmod 5$.

Let $RHSp$ be the number of times p appears on the RHS. Clearly $LHSp \equiv 0 \pmod 5$.

Since $LHSp = RHSp$, this is a contradiction.

END OF SOLUTION

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2. (a) Fill in the following:

0) $0^4 \equiv \qquad \pmod 8$.

1) $1^4 \equiv \qquad \pmod 8$.

2) $2^4 \equiv \qquad \pmod 8$.

3) $3^4 \equiv \qquad \pmod 8$.

4) $4^4 \equiv \qquad \pmod 8$.

5) $5^4 \equiv \qquad \pmod 8$.

6) $6^4 \equiv \qquad \pmod 8$.

7) $7^4 \equiv \qquad \pmod 8$.

(b) Show that there exists an infinite number of n such that n cannot be written as the sum of 6 fourth powers. (HINT: Use Part a.)

SOLUTION

1)

0) $0^4 \equiv 0 \pmod 8$.

1) $1^4 \equiv 1 \pmod 8$.

2) $2^4 \equiv 0 \pmod 8$.

3) $3^4 \equiv 1 \pmod 8$.

4) $4^4 \equiv 0 \pmod 8$.

5) $5^4 \equiv 1 \pmod 8$.

6) $6^4 \equiv 0 \pmod 8$.

7) $7^4 \equiv 1 \pmod 8$.

2) We claim that all numbers $n \equiv 7 \pmod 8$ cannot be written as the sum of 6 fourth powers.

Assume

$$8n + 7 = x_1^4 + \cdots + x_6^4$$

Take both sides mod 8

$$7 \equiv x_1^4 + \cdots + x_6^4 \pmod 8.$$

Each $x_i^4 \pmod 8$ is either 0 or 1. Hence the sum of 6 of them cannot add up to 7.

END OF SOLUTION

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3. (X points) Find a number M such that the following is true, and prove it.

$$(\forall n \geq M)(\exists x, y \in \mathsf{N})[n = 37x + 38y].$$

SOLUTION

Looking ahead we will have two cases: one where $x \geq 1$ so we can swap out a 37-cent, and the other where we swap out SOME number of 38's for SOME number of 37's to get a net gain of 1. So we need to find a, b such that

$37a - 38b = 1$.

Using Wolfram Alpha we find that $a = 37$, $b = 36$ works.

Hence in Case 2 we will need $y \geq 36$, so $n \geq 36 \times 38 = 1368$.

So we will take $M = 1368$.

Base Case: $1368 = 36 \times 38$.

IH: $n \geq 1368$. There exists x, y such that $n - 1 = 37x + 38y$.

IS: Assume there exists $x, y \in \mathsf{N}$ such that $n = 37x + 38y$.

Case 1: $x \geq 1$. Then swap out 1 37-cent coin and swap in one 38-cent coin to get

$$37(x-1) + 38(y+1) = 37x - 37 + 38y + 38 = 37x + 38y + 1 = n + 1.$$

Case 2: $y \geq 36$. Then swap out 36 38-cent coins (thats 1368) and swap in 37 37-cent (thats 1369) to get

$$37(x+37)+38(y-36) = 37x+1369+38y-1368 = 37x+38y+1 = n+1.$$

Case 3: $x \leq 0$ and $y \leq 35$. Then $n \leq 37 \times 0 + 38 \times 35 = 1330$. But $n \geq 1368$. So this case cannot occur.

END OF SOLUTION