Small Ramsey Numbers

Exposition by William Gasarch

April 15, 2022

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

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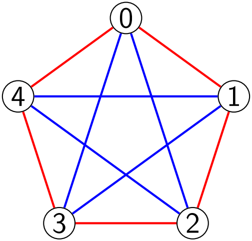
Question What if we color the edges of K_5?

Coloring of K_5 with no Mono K_3



This graph is not arbitrary.
$SQ_5 = \{x^2 \pmod 5) : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then RED.
- If $i - j \notin SQ_5$ then BLUE.

Asymmetric Ramsey Numbers

Definition $R(a,b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a,b) = R(b,a)$.
2. $R(2,b) = b$
3. $R(a,2) = a$

$R(a, b) \leq R(a-1, b) + R(a, b-1)$

Theorem $R(a, b) \leq R(a-1, b) + R(a, b-1)$
Proof
Let $n = R(a-1, b) + R(a, b-1)$. $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$.
Case 1 $(\exists v)[\deg_R(v) \geq R(a-1, b)]$. Look at the $R(a-1, b)$ vertices that are RED to v. By Definition of $R(a-1, b)$ either

- There is a RED K_{a-1}. Combine with v to get RED K_a.
- There is a BLUE K_b.

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Case 3

$(\forall v)[\deg_R(v) \leq R(a-1, b) - 1 \wedge \deg_B(v) \leq R(a, b-1) - 1]$

$(\forall v)[\deg(v) \leq R(a-1, b) + R(a, b-1) - 2 = n - 2]$

Not possible since every vertex of K_n has degree $n - 1$.

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$

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$R(3,4) \leq 9$

Theorem $R(3,4) \leq 9$.
Let COL be a 2-coloring of the edges of K_9.
Case 1 $(\exists v)[\deg_R(v) \geq 4]$. v_1, v_2, v_3, v_4 are RED to v.

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Case 2 $(\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are BLUE to v.
Either:
(1) a RED K_3, or
(2) a BLUE K_3, which together with v is a BLUE K_4.
NOTE Can't have any $\deg_R(v) \leq 2$.

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Case 3 $(\forall v)[\deg_R(v) = 3]$. The RED subgraph has 9 nodes each of degree 3. Impossible!

Reminder of the Odd Vertex Things

Lemma Let $G = (V, E)$ be a graph.

$$V_{\mathrm{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\mathrm{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

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$$\sum_{v \in V_{\mathrm{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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Key: $R(2,4)$ and $R(3,3)$ were both **even!**

Theorem $R(a,b) \leq$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod 2$

Some Better Upper Bounds

- $R(3,3) \le R(2,3) + R(3,2) \le 3 + 3 = 6.$
- $R(3,4) \le R(2,4) + R(3,3) \le 4 + 6 - 1 = 9.$
- $R(3,5) \le R(2,5) + R(3,4) \le 5 + 9 = 14.$
- $R(3,6) \le R(2,6) + R(3,5) \le 6 + 14 - 1 = 19.$
- $R(3,7) \le R(2,7) + R(3,6) \le 7 + 19 = 26$
- $R(4,4) \le R(3,4) + R(4,3) \le 9 + 9 = 18.$
- $R(4,5) \le R(3,5) + R(4,4) \le 14 + 18 - 1 = 31.$
- $R(5,5) \le R(4,5) + R(5,4) = 62.$

Are these tight?

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$COL(a,b) =$ RED if $a - b \equiv SQ \pmod 5$, BLUE OW.

Note $-1 = 2^2 \pmod 5$. Hence $a - b \in SQ$ iff $b - a \in SQ$. So the coloring is well defined.

$R(3,3) \geq 6$

$COL(a,b) =$ RED if $a - b \equiv SQ \pmod{5}$, BLUE OW.

- Squares mod 5: 1,4.
- If there is a RED triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

$$x^2 + y^2 + z^2 \equiv 0 \pmod{5} \text{ Can show impossible}$$

- If there is a BLUE triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

UPSHOT $R(3,3) = 6$ and the coloring used math of interest!

$R(4,4) = 18$

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Vertices are $\{0, \ldots, 16\}$.

Use
$COL(a,b) =$ RED if $a - b \equiv SQ \pmod{17}$, BLUE OW.

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Same idea as above for K_5, but more cases.
UPSHOT $R(4,4) = 18$ and the coloring used math of interest!

$R(3,5) = 14$

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Can we extend these Patterns?

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$R(5,5)$– I will give you a paper to read on that soon.

Revisit those Numbers

Int means Interesting Math. Bor means Boring Math.

- $R(3,3) \le 6$. TIGHT. Int
- $R(3,4) \le 9$. TIGHT. Int
- $R(3,5) \le 14$. TIGHT. Int
- $R(3,6) \le 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3,7) \le 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4,4) \le 18$. TIGHT. Int
- $R(4,5) \le 31$. KNOWN: 25. Both bd Bor
- $R(5,5) \le 62$. KNOWN: Will see it in the paper I give out.

Moral of the Story

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   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.