The Emptier-Filler Game

Consider the following games played between EMPTIER and FILLER. We denote EMPTIER by E and FILLER by F.

1. FILLER fills a box with a finite number of balls, each with a natural number of his choice on it.

2. In every round E takes a ball of his choice from the box. F then counters by replacing the ball with a finite number of other balls, each with a smaller number. (For example E takes a ball labeled 1000, then F replaces it with 999999999 balls labeled 999 and 888876234012 balls labeled 8.)

If the box is ever empty then E wins. If the box is always nonempty (i.e., the game goes on forever) then F wins.

QUESTION: Is there a strategy that E can play so that he will ALWAYS win? Is there a strategy that F can play so that he can ALWAYS win?

VARIANTS:

1. The balls are labeled with integers.

2. The balls are labeled with rationals that are ≥ 0.

3. The balls are labeled with ordered pairs of naturals and the ordering is

$$(0,0) < (0,1) < (0,2) < \cdots (1,0) < (1,1) < (1,2) < \cdots (2,0) < (2,1) < (2,2) < \cdots$$

   (That is, if E removes (i,j) then F can put in as many balls as he wants that are labeled with ordered pairs that are LESS THAN (i,j) in this ordering.)

4. The balls are labeled with ordered triples of natural numbers. Let the ordering be $(a,b,c) < (d,e,f)$ if either (1) $a < d$ or (2) $a = d$ and $b < e$, or (3) $a = d$ and $b = e$ but $c < f$.

QUESTION: Let X be a set and $\preceq$ be an ordering on it. Let the $(X,\preceq)$-game be the game as above where we label the balls with elements from A.

In the following sentence fill in the ???

"E can win the $(X,\preceq)$-game if and only if $(X,\preceq)$ has property ???."