Strong Induction and Inequalities

Nice Recurrences

In the Strong Induction Slide Packet we studied the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 8 & \text{if } n = 1 \\ a_{n-1} + 2a_{n-2} & \text{if } n \geq 2 \end{cases} \tag{1}$$

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$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ a_{n-1} + 11a_{n-2} + 13a_{n-3} & \text{if } n \geq 2 \end{cases} \tag{3}$$

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The answer is
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Bill and Emily both thing that the exact form is gross and not very informative.
We will prove an UPPER BOUND that IS nice.

The Grossest Mathematical Formula In This Course

$$\alpha = (226 - 6\sqrt{327})^{1/3},\ \beta = (2(113 + 3\sqrt{327})^{1/3},\ c_1, c_2, c_3 \in \mathbb{C}.$$

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$g(n) =$

$$c_1\left(\frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})\beta - \frac{(1 - i\sqrt{3})\alpha}{3 \times 2^{2/3}}\right)^n +$$

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Gross and not enlightenting.

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- Knowing an **upper bound** on $g(n)$ is enlightening.

Upper Bound

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Thm $(\forall n)[a_n \leq 5^n]$

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$a_{n-1} \leq 5^{n-1}$,

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Finish on next slide.

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Recall $a_n = a_{n-1} + 11a_{n-2} + 13a_{n-3}$.

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We WANT this to be $\le 5^n$. Lets see:

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$$63 \le 125 \text{ TRUE!}$$

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(5) This is called **Constructive Induction**. It's the topic of the next slide packet.