Quantifiers

250H

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- We can define many different quantifiers such as
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  - There are no more than three
  - There are at least 100
- We also have the ***Uniqueness Quantifier***
  - “There exists a unique x such that P (x) is true.”
    - There is exactly one
    - There is one and only one.

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- Other ways of saying for all or for every
  - all of
  - for each
  - given any
  - for arbitrary
  - for each

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Example: Universal and Existential Quantification

- Determine the truth value of each of these statements if the domain consists of integers (..., -2, -1, 0, 1, 2, ...)
  - $\forall n \, (n + 9 > n)$:
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- Determine the truth value of each of these statements if the domain consists of integers (..., -2, -1, 0, 1, 2, ...)
  - $\exists n\ (2n = 3n)$: True ($n=0$)
  - $\exists n\ (n^2 + 1 = -n)$: False

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Statement	When True?	When False?
∀xP (x)	P (x) is true for every x	There is an x for which P (x) is false.
∃xP (x)	There is an x for which P (x) is true	P (x) is false for every x

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$\exists x P(x)$	There is an x for which P (x) is true	P (x) is false for every x

Precedence

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus
  - $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.
    - it means $(\forall x P(x)) \vee Q(x)$ **NOT** $\forall x(P(x) \vee Q(x))$

Negating Quantified Expressions

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, P (x) is false	There is an x for which P (x) is true
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which P (x) is false	P (x) is true for every x

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    - One quantifier is within the scope of another
        - $\forall x \exists y(x + y = 0)$
        - $\forall x \exists y(x + y = 0) = \forall x Q(x)$
            - $Q(x) = \exists y P(x, y)$
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- Logic to English:
  - $\forall x \forall y(x + y = y + x)$
    - $x + y = y + x$ for all real numbers x and y
  - $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
    - $x + (y + z) = (x + y) + z$ for all real numbers x, y, and z

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  - $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
    - $x + (y + z) = (x + y) + z$ for all real numbers x, y, and z
- It might be helpful to think of this like a nested loop
  - $\forall x \exists y P(x, y)$
  - Loop through the values for x
  - For each x we loop through the values for y

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  - No matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which $P(x, y)$ is true
    - $\forall x \exists y P(x, y)$: y can depend on x
    - $\exists y \forall x P(x, y)$: y is a constant independent of x

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Statement	When True?	When False?
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	P (x, y) is true for every pair x, y	There is a pair x, y for which P (x, y) is false
$\forall x \exists y P(x, y)$	For every x there is a y for which P (x, y) is true	There is an x such that P (x, y) is false for every y
$\exists x \forall y P(x, y)$	There is an x for which P (x, y) is true for every y	For every x there is a y for which P (x, y) is false
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which P (x, y) is true	P (x, y) is false for every pair x, y