# How to Write Proofs

250H

# What is the point of a proof?

- Prove that a statement is true clearly and without **ambiguity**

# Types of Proofs

- **Direct**
  - p → q
  - Assume p
  - Show q

| p | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
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    - $p \rightarrow q$
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- **Contradiction**
    - $p \rightarrow \neg q$
    - Assume p and $\neg q$
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- **Contradiction**
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  - Assume p and $\neg q$
  - Show something goes wrong
- **Contrapositive**
  - $\neg q \rightarrow \neg p$
  - Assume $\neg q$
  - Show $\neg p$

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|---|---|---|
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# Tips on how to start a proof

- What do we know
- What  do we want to show
- What definitions might we need
- What type of proof are we going to use

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  - Def of odd: n is odd if n = 2k + 1 where k is an integer

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- What type of proof are we going to use
  - Direct? No
  - Contradiction? Possibly
  - Contrapositive? Possibly

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  - Let $n \in Z$.
  - For the sake of contradiction, assume $n^2$ is even and n is odd.

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- Use Definitions:
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- Do the algebra:
  - Then, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

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- Finish it:
  - Thus, if $n^2$ is even, then n is even.

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Proof:

Let $n \in \mathbb{Z}$. For the sake of contradiction, assume $n^2$ is even and n is odd. If n is odd then $n = 2k+1$ where k is an integer by the definition of an odd number. Then,

$$n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

Hence we have a contradiction as $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer. Thus, if $n^2$ is even, then n is even. ☽

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  - Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- Spell out your result:
  - Hence $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer.

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- Do the algebra:
  - Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- Spell out your result:
  - Hence $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer.
- Finish it:
  - So, if n is odd, then $n^2$ is odd.
  - Thus, if $n^2$ is even, then n is even.

# Example: Let $n \in \mathbb{Z}$. Prove that if $n^2$ is even, then n is even.

Proof:

Let $n \in \mathbb{Z}$. Assume by way of contrapositive that n is odd.If n is odd then $n = 2k+1$ where k is an integer by the definition of an odd number. Then,

$$n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

Hence, $2(2k^2 + 2k) + 1$ is odd since $2k^2 + 2k$ is an integer. So, if n is odd, then $n^2$ is odd. Thus, if $n^2$ is even, then n is even. ☽

# Tips

- Do not assume your reader knows all definitions
- Do not assume your reader sees what you see
  - It is clear that blah blah blah
  - No it is not
- Do not make things complicated for your reader. It does not make you look more intelligent.