Horse Numbers
Definition of Horse Numbers

**Recall** If \( n \) horses run in a race then the number of ways they can finish is \( n! \).
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**Not Quite**

1. Two horses: \( x_1 < x_2 \) or \( x_2 < x_1 \) or \( x_1 = x_2 \). \( H(2) = 3 \).

2. Three horses. Work on it!

Answer on next slide.
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Not Quite Horses can tie.
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**Due** $H(n)$ is the number of ways $n$ horse can finish a race.
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**Examples** Horses are $x_1, x_2$.

1. 2 horses: $x_1 < x_2$ OR $x_2 < x_1$ OR $x_1 = x_2$. $H(2) = 3$. 

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Three Horses
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\[ x_1 < x_2 < x_3 \quad x_1 < x_3 < x_2 \quad x_1 < x_2 = x_3 \]
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\[ x_1 < x_2 < x_3 \quad x_1 < x_3 < x_2 \quad x_1 < x_2 = x_3 \]

\[ x_2 < x_1 < x_3 \quad x_2 < x_3 < x_1 \quad x_2 < x_3 = x_1 \]
Three Horses

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$x_2 < x_1 < x_3 \quad x_2 < x_3 < x_1 \quad x_2 < x_3 = x_1$

$x_3 < x_1 < x_2 \quad x_3 < x_2 < x_1 \quad x_3 < x_2 = x_1$
Three Horses

\[
\begin{align*}
    x_1 < x_2 < x_3 & \quad x_1 < x_3 < x_2 & \quad x_1 < x_2 = x_3 \\
    x_2 < x_1 < x_3 & \quad x_2 < x_3 < x_1 & \quad x_2 < x_3 = x_1 \\
    x_3 < x_1 < x_2 & \quad x_3 < x_2 < x_1 & \quad x_3 < x_2 = x_1 \\
    x_1 = x_2 < x_3 & \quad x_1 = x_3 < x_2 & \quad x_2 = x_3 < x_1
\end{align*}
\]
Three Horses

\[ x_1 < x_2 < x_3 \quad x_1 < x_3 < x_2 \quad x_1 < x_2 = x_3 \]
\[ x_2 < x_1 < x_3 \quad x_2 < x_3 < x_1 \quad x_2 < x_3 = x_1 \]
\[ x_3 < x_1 < x_2 \quad x_3 < x_2 < x_1 \quad x_3 < x_2 = x_1 \]
\[ x_1 = x_2 < x_3 \quad x_1 = x_3 < x_2 \quad x_2 = x_3 < x_1 \]
\[ x_1 = x_2 = x_3 \]
Three Horses

\[\begin{align*}
    &x_1 < x_2 < x_3 & x_1 < x_3 < x_2 & x_1 < x_2 = x_3 \\
    &x_2 < x_1 < x_3 & x_2 < x_3 < x_1 & x_2 < x_3 = x_1 \\
    &x_3 < x_1 < x_2 & x_3 < x_2 < x_1 & x_3 < x_2 = x_1 \\
    &x_1 = x_2 < x_3 & x_1 = x_3 < x_2 & x_2 = x_3 < x_1 \\
    &x_1 = x_2 = x_3 \\
    H(3) &= 13
\end{align*}\]
Work on it
Four Horses

Work on it
Answer on next slide.
Four Horses: Answer in a Way That Can Generalize

1. Pick one of $x_1$, $x_2$, $x_3$, $x_4$ to be unique min: $4_1$. Order the 3 horses left: $H_3$. Total: $4_1 H_3$

2. Pick two of $x_1$, $x_2$, $x_3$, $x_4$ to be only mins: $4_2$. Order the 2 horses left: $H_2$. Total: $4_2 H_2$

3. Pick three of $x_1$, $x_2$, $x_3$, $x_4$ to be only mins: $4_3$. Order the 1 horse left: $H_1$. Total: $4_3 H_1$

4. Pick three of $x_1$, $x_2$, $x_3$, $x_4$ to be only mins: $4_4$. Order the 0 horses left: $H_0$. Total: $4_4 H_0$

Total: $4_1 H_3 + 4_2 H_2 + 4_3 H_1 + 4_4 H_0 = 75$

Can write in a nicer way for summations:

$4_0 H_0 + 4_1 H_1 + 4_2 H_2 + 4_3 H_3 = 75$
Four Horses: Answer in a Way That Can Generalize

1. Pick one of $x_1, x_2, x_3, x_4$ to be \textbf{unique min}: $\binom{4}{1}$. Order the 3 horses left: $H(3)$. Total: $\binom{4}{1} H(3)$
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3. Pick three of $x_1, x_2, x_3, x_4$ to be only mins: $\binom{4}{3}$. Order the 1 horse left: $H(1)$. Total: $\binom{4}{3}H(1)$

Total: $\binom{4}{0}H(0) + \binom{4}{1}H(3) + \binom{4}{2}H(2) + \binom{4}{3}H(1) = 75$
Four Horses: Answer in a Way That Can Generalize

1. Pick one of \( x_1, x_2, x_3, x_4 \) to be **unique min**: \( \binom{4}{1} \). Order the 3 horses left: \( H(3) \). Total: \( \binom{4}{1} H(3) \)

2. Pick two of \( x_1, x_2, x_3, x_4 \) to be **only mins**: \( \binom{4}{2} \). Order the 2 horses left: \( H(2) \). Total: \( \binom{4}{2} H(2) \)

3. Pick three of \( x_1, x_2, x_3, x_4 \) to be **only mins**: \( \binom{4}{3} \). Order the 1 horse left: \( H(1) \). Total: \( \binom{4}{3} H(1) \)

4. Pick three of \( x_1, x_2, x_3, x_4 \) to be **only mins**: \( \binom{4}{4} \). Order the 0 horses left: \( H(0) \). Total: \( \binom{4}{4} H(0) \)

Total: \( \binom{4}{0} H(0) + \binom{4}{1} H(3) + \binom{4}{2} H(2) + \binom{4}{3} H(1) = 75 \)
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Total:
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Total:

$$\binom{4}{1} H(3) + \binom{4}{2} H(2) + \binom{4}{3} H(1) + \binom{4}{4} H(0) = 75$$
**Four Horses: Answer in a Way That Can Generalize**

1. Pick one of $x_1, x_2, x_3, x_4$ to be **unique min**: $(\begin{pmatrix} 4 \\ 1 \end{pmatrix})$. Order the 3 horses left: $H(3)$. Total: $(\begin{pmatrix} 4 \\ 1 \end{pmatrix})H(3)$

2. Pick two of $x_1, x_2, x_3, x_4$ to be **only mins**: $(\begin{pmatrix} 4 \\ 2 \end{pmatrix})$. Order the 2 horses left: $H(2)$. Total: $(\begin{pmatrix} 4 \\ 2 \end{pmatrix})H(2)$

3. Pick three of $x_1, x_2, x_3, x_4$ to be **only mins**: $(\begin{pmatrix} 4 \\ 3 \end{pmatrix})$. Order the 1 horse left: $H(1)$. Total: $(\begin{pmatrix} 4 \\ 3 \end{pmatrix})H(1)$

4. Pick three of $x_1, x_2, x_3, x_4$ to be **only mins**: $(\begin{pmatrix} 4 \\ 4 \end{pmatrix})$. Order the 0 horses left: $H(0)$. Total: $(\begin{pmatrix} 4 \\ 4 \end{pmatrix})H(0)$

Total:

$$(\begin{pmatrix} 4 \\ 1 \end{pmatrix})H(3) + (\begin{pmatrix} 4 \\ 2 \end{pmatrix})H(2) + (\begin{pmatrix} 4 \\ 3 \end{pmatrix})H(1) + (\begin{pmatrix} 4 \\ 4 \end{pmatrix})H(0) = 75$$

Can write in a nicer way for summations:
Four Horses: Answer in a Way That Can Generalize

1. Pick one of \(x_1, x_2, x_3, x_4\) to be unique min: \(\binom{4}{1}\). Order the 3 horses left: \(H(3)\). Total: \(\binom{4}{1}H(3)\)

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Total:

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\binom{4}{1}H(3) + \binom{4}{2}H(2) + \binom{4}{3}H(1) + \binom{4}{4}H(0) = 75
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For $1 \leq i \leq n$, choose $i$ horses to be the \textit{only min}:

$$H(n) = \sum_{i=1}^{n} \binom{n}{i} H(n-i).$$
$n$ Horses

$H(n)$:
For $1 \leq i \leq n$, choose $i$ horses to be the only min: $\binom{n}{i} = \binom{n}{n-i}$. 
$n$ Horses

$H(n)$:
For $1 \leq i \leq n$, choose $i$ horses to be the only min: \( \binom{n}{i} = \binom{n}{n-i} \).
Order the remaining $n - i$ horses: $H(n - i)$. 
$n$ Horses

$H(n)$:
For $1 \leq i \leq n$, choose $i$ horses to be the \textbf{only min}: $\binom{n}{i} = \binom{n}{n-i}$.
Order the remaining $n - i$ horses: $H(n-i)$.

\[
H(n) = \sum_{i=1}^{n} \binom{n}{n-i} H(n-i) = \sum_{i=0}^{n-1} \binom{n}{i} H(i).
\]