The Emptier-Filler Game
The Players and the Goal

We describe several games between

E: The Emptier
F: The Filler.

▶ If the bin is ever empty then E wins.
▶ If game goes forever and bin is always nonempty then F wins.
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We describe several games between
E: The Emptier
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There will be a bin with numbers in it.
- If the bin is ever empty then E wins.
- If game goes forever and bin is always nonempty then F wins.
1) F puts a finite multiset of \( \mathbb{N} \) into the bin. (e.g., bin has \{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}.)
The Emptier-Filler Game on $\mathbb{N}$

1) F puts a **finite** multiset of $\mathbb{N}$ into the bin. (e.g., bin has $\{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}$.

2) E takes out ONE number $r$ (e.g., 18).
1) F puts a finite multiset of $\mathbb{N}$ into the bin. (e.g., bin has $\{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}$.
2) E takes out ONE number $r$ (e.g., 18).
3) F puts in as many numbers as he wants that are $< r$ (e.g., replace 18 with 99,999,999 17’s and 5000 16’s.)
The Emptier-Filler Game on $\mathbb{N}$

1) $F$ puts a **finite** multiset of $\mathbb{N}$ into the bin. (e.g., bin has \{1, 1, 1, 2, 3, 4, 9, 9, 18, 18\}.

2) $E$ takes out ONE number $r$ (e.g., 18).

3) $F$ puts in **as many numbers as he wants that are** $< r$ (e.g., replace 18 with 99,999,999 17’s and 5000 16’s.)

Which player has the winning strategy? What is that strategy?

**WORK IN GROUPS!**
E wins!

Strategy for E

Keep removing the largest number in the box.

Why does this work?
The highest rank in the bin is $r$.
There are $n$ balls of rank $r$.
After the first $n$ moves of E the highest ranking number is $\leq r - 1$.
The (MANY) balls in the bin all have rank $\leq r - 1$.
Keep doing this. Eventually they all have rank 0 and when removed cannot be replaced.

This proof is not rigorous. We return to this point later.
Answer!

E wins!

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The highest rank in the bin is \( r \).
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Answer!

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The (MANY) balls in the bin all have rank $\leq r - 1$.

Keep doing this. Eventually they all have rank 0 and when removed cannot be replaced.

This proof is not rigorous. We return to this point later.
The Emptier-Filler Game on Other Orderings

**WORK IN GROUPS** Determine who wins the Emptier-Filler Game on these orderings:

1. \( \mathbb{Z} \)
2. \( \mathbb{Q}_{\geq 0} \)
3. \( \mathbb{Q}_{> 0} \)
4. \( \mathbb{N} + \mathbb{N} \)
5. \( \mathbb{N} + \mathbb{N} + \cdots + \mathbb{N} \) 100 times.
6. \( \mathbb{N} + \mathbb{N} + \mathbb{N} + \cdots \) goes on forever
Answers!

1. Z:

2. $Q \geq 0$ and $Q > 0$:

3. $N + N$:

4. $N + N + N + \ldots$ 100 times:

5. $N + N + N + \ldots$ goes on forever:
Answers!

1. \( \mathbb{Z} \): F wins.
Answers!

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2. \(\mathbb{Q}_{\geq 0}\) and \(\mathbb{Q}_{> 0}\):
Answers!

1. $\mathbb{Z}$: F wins. If E removes $x$, F puts in $x - 1$.
2. $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{> 0}$: F wins.
Answers!

1. \( \mathbb{Z} \): F wins. If E removes \( x \), F puts in \( x - 1 \).
2. \( Q_{\geq 0} \) and \( Q^{>0} \): F wins. If E removes \( x \), F puts in \( \frac{x}{2} \).
Answers!

1. \( \mathbb{Z} \): F wins. If E removes \( x \), F puts in \( x - 1 \).
2. \( \mathbb{Q}_{\geq 0} \) and \( \mathbb{Q}^{> 0} \): F wins. If E removes \( x \), F puts in \( \frac{x}{2} \).
3. \( \mathbb{N} + \mathbb{N} \):
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1. \( \mathbb{Z} \): F wins. If E removes \( x \), F puts in \( x - 1 \).
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Answers!

1. $\mathbb{Z}$: F wins. If E removes $x$, F puts in $x - 1$.
2. $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{> 0}$: F wins. If E removes $x$, F puts in $\frac{x}{2}$.
3. $\mathbb{N} + \mathbb{N}$: E wins.
   
   **Key** Keep Remove the largest ball in second $\mathbb{N}$.
Answers!

1. \( \mathbb{Z} \): F wins. If E removes \( x \), F puts in \( x - 1 \).
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3. \( \mathbb{N} + \mathbb{N} \): E wins.
   - **Key** Keep Remove the largest ball in second \( \mathbb{N} \).
   - Eventually the second \( \mathbb{N} \) balls are all 0’s.

4. \( \mathbb{N} + \mathbb{N} + \mathbb{N} + \cdots \) goes on forever: E wins.
   - **Key** When filler initially puts balls in the bin he only uses (say) the first 100,000 \( \mathbb{N} \)’s. At that point you are playing the game with \( \mathbb{N} + \mathbb{N} + \cdots \) (100,000 times).
Answers!

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   **Key** Keep Remove the largest ball in second $\mathbb{N}$.
   Eventually the second $\mathbb{N}$ balls are all 0’s.
   When these are removed then $F$ must replace them with some elements of the first $\mathbb{N}$.

4. $\mathbb{N} + \mathbb{N} + \mathbb{N} + \cdots$ goes on forever:
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4. $\mathbb{N} + \mathbb{N} + \cdots + \mathbb{N}$ 100 times.
Answers!

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Answers!

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   **Key** When filler initially puts balls in the bin he only uses (say) the first 100,000 $\mathbb{N}$’s.
   At that point you are playing the game with $\mathbb{N} + \cdots + \mathbb{N}$ (100,000 times).
Question Let $X$ be a set and $\preceq$ be an ordering on it. Let the $(X, \preceq)$-game be the game as above where we put elements of $X$ in the bin.
Need a General Theorem

**Question** Let $X$ be a set and $\preceq$ be an ordering on it. Let the $(X, \preceq)$-game be the game as above where we put elements of $X$ in the bin.

In the following sentence fill in the BLANK.

E can win the $(X, \preceq)$-game if and only if $(X, \preceq)$ BLANK.

WORK IN GROUPS!
**Def** $(X, \preceq)$ is **well ordered** if there are NO infinite decreasing sequences.
Def \((X, \leq)\) is well ordered if there are NO infinite decreasing sequences.

E can win the \((X, \leq)\)-game if and only if \((X, \leq)\) is well ordered.
Rigorous Proof that E wins on $\mathbb{N}$

Strategy for E
Keep removing the largest number in the box.

Why does this work?
Let's prove it by induction! But on what?

1) Ind on number of balls.
NO GOOD—it often goes UP!

2) Ind on highest ranked ball.
NO GOOD—it often stays the same.

3) So what to do induction on? Discuss
Answer on next slide.
Rigorous Proof that E wins on $\mathbb{N}$

Strategy for E: Keep removing the largest number in the box.
Rigorous Proof that E wins on \( \mathbb{N} \)

**Strategy for E** Keep removing the largest number in the box.

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Rigorous Proof that E wins on $\mathbb{N}$

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Rigorous Proof that E wins on \( \mathbb{N} \)

**Strategy for E** Keep removing the largest number in the box.

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3) So what to do induction on? Discuss
Rigorous Proof that $E$ wins on $\mathbb{N}$

**Strategy for $E$** Keep removing the largest number in the box.

**Why does this work?** Lets prove it by induction! But on what?

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3) So what to do induction on? Discuss Answer on next slide.
Ind on a Funky Ordering

Assume that the highest rank of a ball is $r$.
Assume that the number of balls of rank $r$ is $n$.
Then we associate to the position the ordered pair $(r, n)$.
Consider the following funky ordering on ordered pairs.
$$(0, 0) < (0, 1) < (0, 2) < \cdots < (1, 0) < (1, 1) < (1, 2) < \cdots$$
This is the ordering to use.
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This is the ordering to use.
Formal Proof

**Thm** E wins the game by removing the largest ranked ball.
Formal Proof

Thm E wins the game by removing the largest ranked ball.

Proof By induction on the funky ordering.

\[ (0,0) \] E has already won.

IH Assume that for all \((n', r') < (n, r)\), E wins.

IS The game is at position \((n, r)\).

If E removes the top ranked ball and F puts in as many balls of lower rank then

\[ \text{If there were } \geq 2 \text{ balls of rank } r \text{ then new position is } (n-1, r) < (n, r). \]

\[ \text{If there was 1 ball of rank } r \text{ then the new position is } (n', r') \text{ where } r' \leq r - 1. \text{ Note } (n', r') < (n, r). \]

From here, by the IH, E wins.

End of Proof
Thm E wins the game by removing the largest ranked ball.

Proof By induction on the funky ordering.

B (0, 0). E has already won.

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Formal Proof

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**End of Proof**
The Funky Ordering

\[(0, 0) < (0, 1) < (0, 2) < \cdots < (1, 0) < (1, 1) < (1, 2) < \cdots \cdots \].
The Funky Ordering

\[(0, 0) < (0, 1) < (0, 2) < \cdots < (1, 0) < (1, 1) < (1, 2) < \cdots \cdots \]

But Can we do induction on this funky ordering?
The Funky Ordering

\[(0, 0) < (0, 1) < (0, 2) < \cdots < (1, 0) < (1, 1) < (1, 2) < \cdots \ldots \cdot\]

**But** Can we do induction on this funky ordering?
Yes- Its a well ordering.
The Funky Ordering

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If you start at \((n, r)\) and march downward will you get to \((0, 0)\) in a finite number of steps? Discuss.
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But that does not matter.

**Def** An ordering is **well ordered** if when you start at any element \(x\) and march downward you will get to a MIN element in a finite number of steps.
The Funky Ordering

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Yes.
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But that does not matter.

**Def** An ordering is **well ordered** if when you start at any element \(x\) and march downward you will get to a MIN element in a finite number of steps.

**Upshot** You can do induction on any well ordered ordering.
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank.
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank. What if E removes a ball of lowest rank? Does She still win? Vote

No matter what E does, she wins! How to prove that? By an induction on a funkier ordering. We won't be doing that.
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank. What if E removes a ball of lowest rank? Does She still win? \textbf{Vote YES.}
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank. What if E removes a ball of lowest rank? Does She still win? Vote YES.

What if E and F both want E to lose? Is their a strategy for both of them to make this happen?
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank. What if E removes a ball of lowest rank? Does She still win? Vote YES.

What if E and F both want E to lose? Is their a strategy for both of them to make this happen? NO.

No matter what E does, she wins!
Does the Strategy Matter?

Our strategy was that E always removes a ball of highest rank. What if E removes a ball of lowest rank? Does She still win? **Vote YES.**

What if E and F both want E to lose? Is their a strategy for both of them to make this happen? **NO.**

**No matter what E does, she wins!**

**How to prove that?** By an induction on a an even funkier ordering. We won’t be doing that.
Let \((X, \leq)\) be a well ordering. Then no matter how E and F play the game, E will win.
General Theorem

Let \((X, \leq)\) be a well ordering. Then no matter how E and F play the game, E will win. Even if she doesn’t want to.
General Theorem

Let \((X, \leq)\) be a well ordering. Then no matter how E and F play the game, E will win. Even if she doesn’t want to.

Let \((X, \leq)\) be a NOT a well ordering then F has a winning strategy.
General Theorem

Let \( (X, \leq) \) be a well ordering. Then no matter how E and F play the game, E will win. Even if she doesn’t want to.

Let \( (X, \leq) \) be a NOT a well ordering then F has a winning strategy. We leave this to you.
General Theorem

Let \((X, \leq)\) be a well ordering. Then no matter how E and F play the game, E will win.
Even if she doesn’t want to.

Let \((X, \leq)\) be a NOT a well ordering then F has a winning strategy.
We leave this to you.
Can E and F play so that F loses. Yes- F just never puts anything in the bin.