Predicate and Quantifier Review

250H
Negating Quantified Expressions

<table>
<thead>
<tr>
<th>Negation</th>
<th>Equivalent Statement</th>
<th>When Is Negation True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬ ∃ x P (x)</td>
<td>∀ x ¬ P (x)</td>
<td>For every x, P (x) is false</td>
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</table>
The Order of Quantifiers

- Order Matters
  - Unless all quantifiers are universal quantifiers or all are existential quantifiers
- The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent
  - The statement $\exists y \forall x P(x, y)$ is true if and only if there is a $y$ that makes $P(x, y)$ true for every $x$.
  - There must be a particular value of $y$ for which $P(x, y)$ is true regardless of the choice of $x$.
  - $\forall x \exists y P(x, y)$ is true if and only if for every value of $x$ there is a value of $y$ for which $P(x, y)$ is true
  - No matter which $x$ you choose, there must be a value of $y$ (possibly depending on the $x$ you choose) for which $P(x, y)$ is true
    - $\forall x \exists y P(x, y)$: $y$ can depend on $x$
    - $\exists y \forall x P(x, y)$: $y$ is a constant independent of $x$
Logical Operator: Conditional Statements

Common ways to express $p \rightarrow q$:

- if $p$, then $q$
- $p$ implies $q$
- if $p$, $q$
- $p$ only if $q$
- $p$ is sufficient for $q$
- a sufficient condition for $q$ is $p$
- $q$ if $p$
- $q$ whenever $p$
- $q$ when $p$
- $q$ is necessary for $p$
- a necessary condition for $p$ is $q$
- $q$ follows from $p$
- $q$ unless $\neg p$
Example 1: Translating Math Statements into Statements

- Translate the statement “The sum of two positive integers is always positive” into a logical expression
  - Rewrite it so that the implied quantifiers and a domain are shown
    - For every two integers, if these integers are both positive, then the sum of these integers is positive.
  - Introduce the variables $x$ and $y$ to obtain
    - For all positive integers $x$ and $y$, $x + y$ is positive.
  - Quantify
    - $\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$, where the domain for both variables consists of all integers.
    - Alternate Solution: $\forall x \forall y (x + y > 0)$, where the domain for both variables consists of all positive integers.
Example 2: Translating Math Statements into Statements

- Translate the statement: Every real number except zero has a multiplicative inverse. 
  \(\text{(A multiplicative inverse of a real number } x \text{ is a real number } y \text{ such that } xy = 1.)\)
  - Rewrite it so that the implied quantifiers and a domain are shown
    - For every real number \(x\) except zero, \(x\) has a multiplicative inverse.
  - Introduce the variables \(x\) and \(y\) to obtain
    - For every real number \(x\), if \(x \neq 0\), then there exists a real number \(y\) such that \(xy = 1\)
  - Quantify
    - \(\forall x((x \neq 0) \rightarrow \exists y(xy = 1))\)
Example 3: Translating Math Statements into Statements

- Translate the statement: There exists two distinct rational numbers such that $xy = 0$.
  - $\exists x, y \in \mathbb{Q} ((x \neq y) \land (xy = 0))$
Example 4: Translating Math Statements into Statements

- Translate the statement: There exists an infinite number of natural numbers.
  - $\forall x \in \mathbb{N} \, \exists y \in \mathbb{N} \, (y > x)$
Example 5: Translating Math Statements into Statements

- Translate the statement: There are no natural numbers x, y such that xy = -1.
  - ¬( ∃ x,y (xy = -1))
  - ∀ x, y ∈ N (xy ≠ -1)