Homework 02, MORALLY Due Feb 17

1. (40 points) For this problem FIRST write the program and make conjectures THEN look up whats true.

   (a) (0 points) Write a program that does the following: For every $1 \le n \le 1000$:

   - Using the greedy method (Leo will tell you about that in class) determine natural numbers $(x_1, x_2, \ldots, x_k)$ such that $x_1^3 + \cdots + x_k^3 = n$. Using brute force (Leo will tell you about that in class) determine natural numbers $(x_1, x_2, \ldots, x_L)$ such that $x_1^3 + \cdots + x_L^3 = n$.
   - Below I have the first 9 rows of the output. I use *Gr* for *Greedy* and *Br* for *Brute* to make the picture fit on the page. I only partially succeeded. Oh well.

n	Gr-k	Greedy-x_i's	Br-L	Brute-x_i's
1	1	$1 = 1^3$	1	$1 = 1^3$
2	2	$2 = 1^3 + 1$	2	$2 = 1^3 + 1^3$
3	3	$3 = 1^3 + 1^3 + 1^3$	3	$3 = 1^3 + 1^3 + 1^3$
4	4	$4 = 1^3 + 1^3 + 1^3 + 1^3$	4	$4 = 1^3 + 1^3 + 1^3 + 1^3$
5	5	$5 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	5	$5 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3$
6	6	$6 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	6	$6 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 +$
7	7	$7 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$	7	$7 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 +$
8	1	$8 = 2^3$	1	$8 = 2^3$
9	2	$9 = 2^3 + 1^3$	2	$9 = 2^3 + 1^3$

   DO NOT hand in the program or the output.

GO TO NEXT PAGE FOR THE REST OF THIS PROBLEM

(b) (30 points) Based on this data make a conjectures of the following forms:

   i. Every n is the sum of $\le XXX$ cubes. Write your conjecture in quantifiers.

   ii. All but a finite number of n is the sum of $\le XXX$ cubes. Write your conjecture in quantifiers.

   iii. The Greedy algorithm is optimal XXX. (XXX should be finitely often of infinitely often.)

(c) (10 points) Look on the web and/or use AI to determine what is known and what is conjectured about these problems.

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2. (30 points) In this problem I will give a property of a domain $\mathbb{D}$ and I want you to give me (a) that property expressed as quantifiers, and (b) either an example of that domain or the statement (without proof) that there is no such $\mathbb{D}$. I give two EXAMPLES:

   Example 1: $\mathbb{D}$ has a least element

   Expressed as quantifiers: $(\exists x \in \mathbb{D})(\forall y \in \mathbb{D})[x \le y]$.

   Domain that works: $\mathbb{N}$ or you could write $\{0, 1, 2, \ldots\}$.

   Example 2: $\mathbb{D}$ is an infinite subset of $\{1, 2, 3\}$.

   (We use that if $\mathbb{D}$ is infinite then it either has an infinite increasing sequence or an infinite decreasing sequence.)

   Expressed as quantifiers:

   $$((\forall x \in \mathbb{D})(\exists y \in \mathbb{D})[x < y] \vee (\forall x \in \mathbb{D})(\exists y \in \mathbb{D})[x > y])) \wedge (\forall x \in \mathbb{D})[x \in \{1, 2, 3\}]$$

   There is no such $\mathbb{D}$: Even though its not required I will give a proof: If $\mathbb{D}$ is a subset of $\{1, 2, 3\}$ then $|\mathbb{D}| \le 3$ and hence is finite.

   (a) (10 points) $\mathbb{D}$ is infinite and has both a least element and a greatest element.

   (b) (10 points) Every element $x \in \mathbb{D}$ has both a successor element y (so $x < y$ and there is nothing inbetween) and also a predecessor element w (so $w < x$ and there is nothing inbetween).

   (c) (10 points) $\mathbb{D}$ satisfies both the property in Part a and Part b. For this one you don't need to give the formula since its just the AND of the first two parts. So the question here is to determine if such a $\mathbb{D}$ exists and either give me the $\mathbb{D}$ or prove there is no $\mathbb{D}$.

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3. (30 points) In this problem the only symbols you can use are

- The usual logical ones: $\forall$, $\exists$ $\wedge$, $\vee$, $\neg$.
- Variables that range over the domain: $x_1, x_2, x_3, \ldots$.
- These math symbols: $<$, $\leq$, $>$, $\geq$, $=$, $\neq$.

For each of the following either give me the sentence I want OR state (without proof) that there is no such sentence.

(a) (10 points) A sentence which is true with domain $\mathbb{Z}$ but false with domain $\mathbb{Q}$.

(b) (10 points) A sentence which is true with domain $\mathbb{Z}$ but false with domain $(0, 1)$. (Recall that $(0, 1)$ is the set of reals between 0 and 1 but NOT including 0 or 1.)

(c) (10 points) A sentence which is true with domain $(0, 1)$ but false with domain $\mathbb{Q}$. (Warning: You can't use $(\exists x)[x^2 = \frac{1}{2}]$ since this cannot be stated just using $<$.)