Duplicator Spoiler Games

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We will call SPOIL S and DUP D to fit on slides.

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Bill plays a student $(L_3, L_4, 2)$**,** $(L_3, L_4, 3)$

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Play a student $\mathbb{N}$ and $\mathbb{Z}$ with 1 move, 2 moves

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A Notion of L, L' being Similar

Let L and L' be two linear orderings.

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Let L and L' be two linear orderings.
Def If D wins the k-round DS-game on L, L' then L, L' are **k-game equivalent** (denoted $L \equiv_k^G L'$).

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Upshot We need to define qd rigorously.

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$$\mathrm{qd}((\forall x)(\forall z)[x < z \rightarrow (\exists y)[x < y < z]]) = 2 + 1 = 3$$

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Def L and L' are ***k*-truth-equiv** ($L \equiv_k^T L'$)

$$(\forall \phi, \mathrm{qd}(\phi) \le k)[L \models \phi \text{ iff } L' \models \phi.]$$

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3. Upshot: Questions about expressability become questions about games.
4. Complexity: As Computer Scientists we think of complexity in terms of time or space (e.g., sorting n elements can be done in roughly $n \log n$ comparisons). But how do you measure complexity for concepts where time and space do not apply? One measure is quantifier depth. These games help us prove LOWER BOUNDS on quantifier depth!

Proving DUP Wins Rigorously

Notation

The game where the orders are L and L', and its for n moves, will be denoted

$$(L, L'; n)$$

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DUP respond with x in L_b. DUP views game as 2 GAMES:

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  By IH DUP wins $(L_{a-x}, L_{b-x}; n-1)$.

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3. Might not need induction on the smaller boards if they are orderings we already proved things about.

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IH DUP wins $(\mathbb{N}, \mathbb{N}+\mathbb{Z}; n-1)$.
1) SP plays x in either $\mathbb{N}$ or $\mathbb{N}$-part of $\mathbb{N}+\mathbb{Z}$ then DUP counters with the same x in the other part. The 2 games are

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SP won't play on 1st board.
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2) SP plays x in $\mathbb{Z}$ part of $\mathbb{N}+\mathbb{Z}$ then DUP plays 2^n in $\mathbb{N}$. The 2 games are

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$(\mathbb{N}+\mathbb{N}^*, L_{2^n}; n-1)$ and $(\mathbb{N}, \mathbb{N}; n-1)$.
SP won't play on 2nd board. DUP wins 1st board by prior thm.