

Loaded Dice

Fair Dice Yield Unfair Sums

Fair Die:

$$\Pr(1)=\Pr(2)=\Pr(3)=\Pr(4)=\Pr(5)=\Pr(6) = 1/6 \sim 0.167$$

Roll TWO of them.

$\Pr(\text{Sum}=2)=1/36$ (This is Min $\Pr(\text{Sum})$)

$\Pr(\text{Sum}=7)=1/6$. (This is Max $\Pr(\text{Sum})$)

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How Unfair?: $1/6 - 1/36 \sim 0.139$ unfair.

What are Loaded Dice?

Def A **Die** is a 6-tuple $(p_1, p_2, p_3, p_4, p_5, p_6)$ such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^6 p_i = 1$.

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- 1) There exists a way to load dice so that all sums are prob $\frac{1}{11}$.
- 2) There is no way to load dice so that all the sums are prob $\frac{1}{11}$.

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- 1) There exists a way to load dice so that all sums are prob $\frac{1}{11}$.
 - 2) There is no way to load dice so that all the sums are prob $\frac{1}{11}$.
- No such dice can exist!

Polynomials are our Friends!

Assume that are dice that yield fair sums. Let $(p_1, \dots, p_6)$ and $(q_1, \dots, q_6)$ be those dice.

KEY:

$$(p_1x + p_2x^2 + \dots + p_6x^6)(q_1x + q_2x^2 + \dots + q_6x^6)$$

Coefficient of x^5 is

$$p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1 = \text{Prob}(\text{sum} = 5)$$

Coefficient of x^i is $\text{Prob}(\text{sum} = i)$.

Fair Sums- NOT!

Let $(p_1, \dots, p_6)$ and $(q_1, \dots, q_6)$ be dice. **Assume** they yield FAIR SUMS, all sums have prob $1/11$. Then

$$(p_1x + \dots + p_6x^6)(q_1x + \dots + q_6x^6) = (1/11)(x^2 + x^3 + \dots + x^{12})$$

So

$$(p_1 + \dots + p_6x^5)(q_1 + \dots + q_6x^5) = (1/11)(1 + x + x^2 + \dots + x^{10})$$

Two Polynomials

Recap If $(p_1, \dots, p_6)$ and $(q_1, \dots, q_6)$ are two loaded dice that yield fair sums then:

$$(p_1 + \dots + p_6 x^5)(q_1 + \dots + q_6 x^5) = (1/11)(1 + x + x^2 + \dots + x^{10})$$

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Real Roots of Left Polynomials

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Upshot The Left poly has ≥ 3 real roots.

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Recall Upshot The Left poly has ≥ 3 real roots.

Lets look at the roots of the right poly:

$$x^{11} - 1 = 0$$

$$x^{11} = 1$$

All roots on complex unit circle. Hence ≤ 2 real roots.

Upshot The Right poly has ≤ 2 real roots.

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Upshot The Right poly has ≤ 2 real roots.

Final Upshot The left and right poly DIFFER on the number of real roots, so they cannot be the same. Contradiction!