Some Cool Proofs + Graph Theory

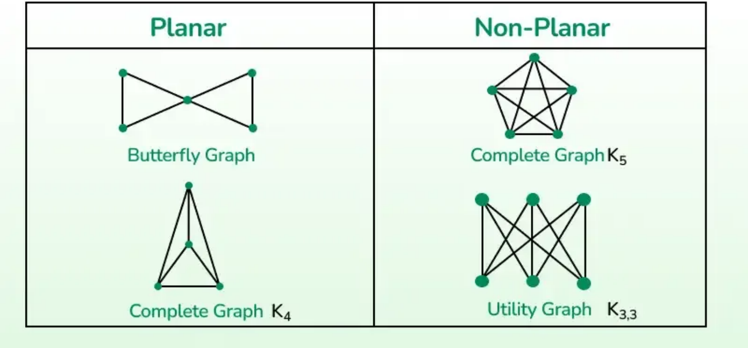
CMSC250H

February 18, 2026

Planar Graphs

Planar Graphs

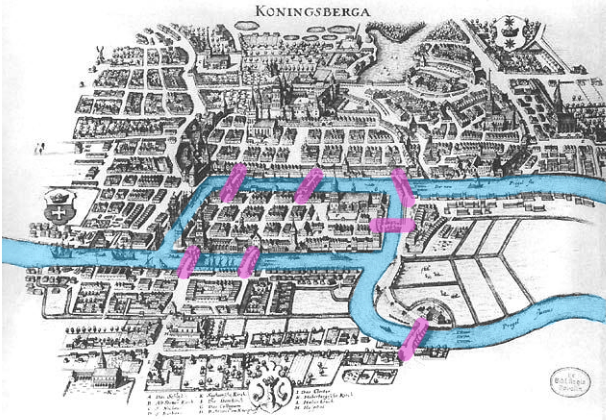
A graph $G = (V, E)$ is planar if an only if it can be drawn on the plane so that no two edges cross.



Historical Motivation

Seven Bridges of Königsberg

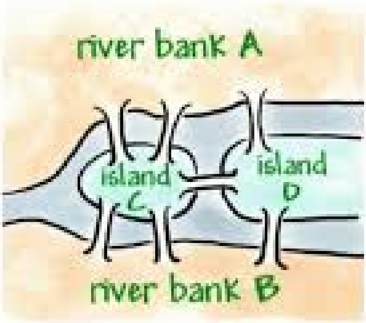
Can we make a walk through the city such that we visit every area, and never cross the same bridge?

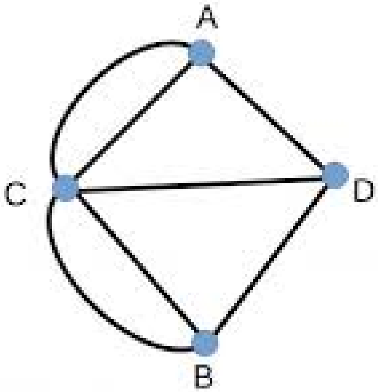


Eulerian Graphs

- The city of Königsberg actually asked Euler to solve this problem.
- He proved that you couldn't and laid the foundation for graph theory
- A graph like this is non-Eulerian
- For a graph to be Eulerian every vertex must have even degree (its connected to an even number of vertices)

Königsberg Graph





Known Fact

For a planar graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ the following is always true

$$m \leq 3n - 6$$

Can think of this as "we don't want the graph having too many connections", or else it won't be planar

Every Planar Graph Has a Vertex $\leq$5 neighbors

Proof.

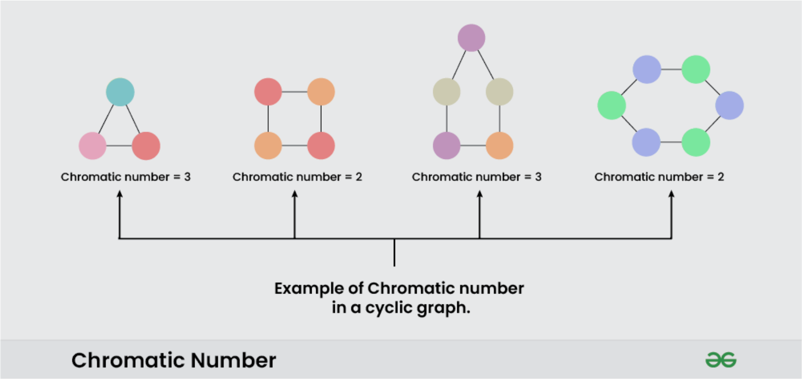
Let's assume that this isn't the case, and all have at least 6 neighbors. Let's count up all the edges in the graph. Suppose $|E| = m$. Then if we count the neighbors of each vertex (including repetitions) we get

$$2m = \sum_{v \in V} \deg(v) \geq 6n \implies m \geq 3n$$

This doesn't work, since we know that for planar graphs $m \leq 3n - 6$, a contradiction. □

Graph Coloring

We say a graph is *k*-colorable if we can color every vertex in the graph using *k* colors, and no two vertices that share an edge are the same color.



5-Color Theorem

5-Color Theorem

Every planar graph can be colored using 5 colors.

Proof of 5-Color Theorem

Proof.

Clearly, we know that a planar graph of size one is k-colorable.

Let's assume that the theorem holds for all planar graphs of size k.

Now, consider a graph G of size $k+1$. What if we delete the vertex from G with the smallest number of neighbors (which we know is less than or equal to 5)? Call this vertex v.

This creates a graph of size k! Therefore, this resulting graph is 5-colorable. □

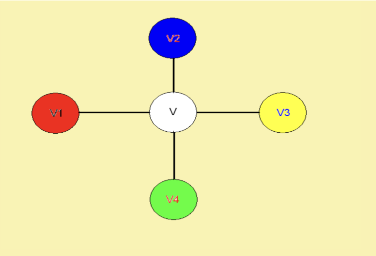
Proof of 5-Color Theorem Cont.

Proof.

Let's see what happens when we add v back. Since we picked to delete the vertex with the smallest degree, it has to have degree less than or equal to 5.

If it has 4 neighbors or less than we can just color the vertex we just added back the 5th color. This makes it 5-colorable.

The remaining case is that it has 5 neighbors. □



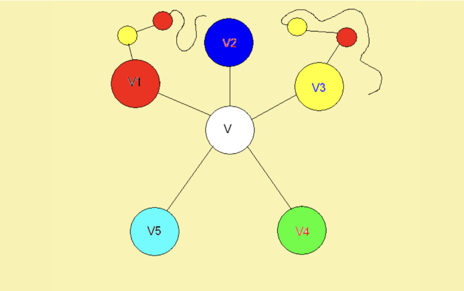
Proof of 5-Color Theorem Cont.

Proof.

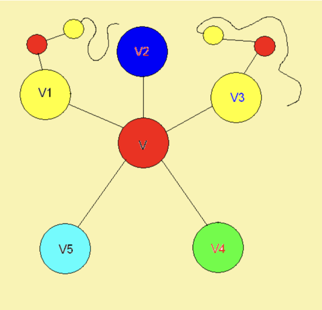
Let's say v has 5 neighbors (and that they are all colored differently). Let's consider two neighbors of v, call them u and w, u being Color 1 and w being Color 2, and then look at all the vertices in the graph that are either Color 1 or Color 2 ($G_{1,3}$). If u and w do not have a path between them (i.e they can't reach each other) then we can switch the color of u to that of w, and do the same for all the other vertices in the component of u.

This continues to make the graph 5-colorable, and now u and w have the same color. Now we can add vertex v back and color it with the remaining color, since two of its neighbors share a color. □

Picture of 5 Neighbor Case



$\Longrightarrow$

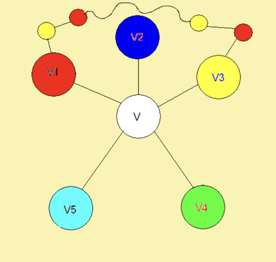


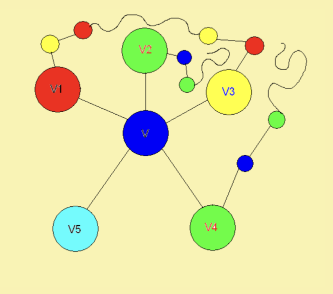
Last Subcase

Proof.

Let's assume that they are in the same component. Let's now look at the other neighbors of v, call them x and y, x being Color 2 and y Color 4. If we consider $G_{2,4}$, and they don't share a path, then we switch the color of x to that of y and follow the same procedure.

Let's assume they are in the same component again. □





Proof.

If they are connected, then there is a path between them. But, uh-oh... look at the picture! This forces the two paths to cross! But this can't be the case because our graph is planar. So, one of the other cases must be true, and we have that G is 5-colorable. □

Induction

- This is an example of induction
- We establish our claim is valid for a base case, which is normally really easy
- Assume it works up to a certain point, then use this assumption to ”construct” and force the next one in the chain to be true
- We induct on the natural numbers
- This proof is nice because it has a really cool visual intuition, however

Are Planar Graphs 4-Colorable?

- Planar Graphs are 4-Colorable
- Planar Graphs are only 5-Colorable

4-Colorable

Indeed they are! It is a much harder proof, so we won't be doing that :)

Why Introduce you to Graphs

- There are a lot of cool proofs involving graphs
- They are an integral part of computer science
- You have probably already seen or heard of them
- They are visually intuitive!
- They are practical and used in a lot of applications!