Some Cool Proofs + Graph Theory

CMSC250H

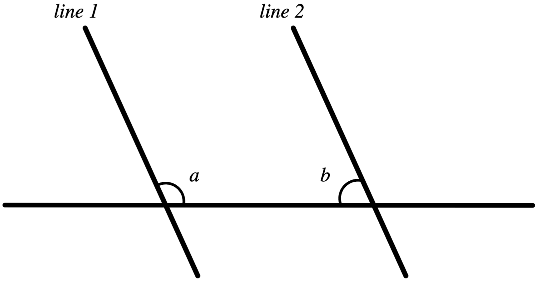
February 18, 2026

Logical Systems

Normally a mathematical or logical system has some fundamental truths to them: we call these axioms.
A famous example is Euclid's axioms for geometry (Euclid's Postulates):

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.
5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles. (Parallel lines never meet and have constant distance)

Parallel Line Postulate



If: $a + b = 180°$

Then: *line 1 and line 2 are parallel*

History of The Parallel Postulate

- This postulate wasn't seen as intuitive to mathematicians
- They wanted to be able to prove this statement using the other 4 axioms, but couldn't so they added it as an axiom
- Spherical Geometry assumes this axiom is false and that there exists no parallel lines

Are there lots of these Statements?

- Sometimes we observe that something is true
- Sometimes it's really hard to prove the statement holds all the time
- Can we somehow find an algorithm to find all truths given a set of axioms and the basic rules of logic?

Definitions of Logical Systems

Completeness

A mathematical system is consistent if we can derive all truths from its axioms.

Consistent

A mathematical system is consistent if we cannot derive two contradictory statements (i.e P and $\neg P$)

Hilbert's Program

Hilbert wanted to axiomatize, express a theory as a set of axioms, all mathematical theories using a finite set of axioms and prove that these axioms were consistent. (Original G.O.A.T Mathematician)

This would, in turn, give us an algorithm for finding all truths in a system.

Gödel's Incompleteness Theorem

Gödel's First Theorem

Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e. there are statements of the language of F which can neither be proved nor disproved in F

Through a technique called Gödel numbering, Gödel encoded arithmetical expressions into unique numbers (prime powers). This allows the formal system to talk about itself since it needs to support basic arithmetic operations. He created a formal mathematical statement that asserts its own unprovability within a specific system, essentially stating, "This sentence is not provable in the system".

Second Theorem

Gödel's Second Theorem

It is not possible to prove whether a set of axioms in a formal system F is consistent.

What does this imply

- Somethings are really hard to prove
- Somethings literally cannot be proven
- Some famous conjectures:
  - Goldbach Conjecutre: Every even number is the sum of 2 primes
  - Riemann Hypothesis: Distribution of primes. Zeros encode the precise fluctuations, or ”harmonics,” in the distribution of prime numbers. All lie on the same line.
  - Twin prime conjecture: There are infinite prime numbers p such at $p+2$ is also prime