Aaron George Pumping Lemma

Exposition by William Gasarch

Here is the standard pumping lemma:

Lemma 0.1 *Let L be regular via DFA M which has s states. Then for all $w \in L \cap \Sigma^{\ge s+1}$ there exists x, y, z such that the following happen:*

- $w = xyz$
- $y \ne e$ *(it could be that $x = e$ or $z = e$),*
- $xy^*z \subseteq L$

This can be used to show that $\{a^n b^n : n \in \mathsf{N}\}$ is NOT regular. We omit this since its any any Formal Lang Textbook and on the web.

What about $L = \{w : \#_a(w) = \#_b(w)\}$? The pumping lemma above cannot be used directly to show L is not regular. We need also use closure properites.

If L is regular than $L \cap a^*b^* = \{a^n b^n : n \in \mathsf{N}\}$ is regular, which we have shown it is not.

But there is another way.

Aaron George Lemma:

Lemma 0.2 *Let L be regular via DFA M which has s states. Then for all $w \in L \cap \Sigma^{\ge s+1}$ there exists x, y, z such that the following happen:*

- $w = xyz$
- $y \ne e$ *(it could be that $x = e$ or $z = e$ or $z' = e$),*
- $xy^*z \subseteq L$
- *(This is whats NEW)* $|xy| \le 2s$.

This really comes out of looking at the proof of the pumping theorem more carefully. $|x| \le n$ since x only visits every state at most one. $|y| \le n$ for the same reason. Hence $|xy| \le 2n$.

This new pumping lemma can be used to show L not regular BUT can also be used to give an EASIER proof that $a^n b^n$ is not regular.

Proof that $\{a^n b^n : n \in \mathsf{N}\}$ is NOT Regular Assume that it is regular via a DFA on s states. Let $n = 2s$. Look at $a^n b^n$. By Aaron George Pumping Lemma

$a^n b^n = xyz$ where $|xy| \le 2s = n$. Hence xy has ONLY a's in it. Hence y' has ONLY a' in it. ONLY one case:

$x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3} b^n$. Only restriction is that

$n_1 + n_2 + n_3 = n$

and

$n_2 \neq 0$.

Since $xyyz$ is in $a^n b^n$ we get

$a^{n_1 + 2n_2 + n_3} b^n \in a^n b^n$

We leave it to the reader to get the contradiction.

The proof for $\{w : \#_a(w) = \#_b(w)\}$ is not regular is... IDENTICAL! The statement of the Aaron George pumping lemma is

For ALL w blah blah. We'll just take $w = a^n b^n$. The only case we need has to do with $\#_a(w)$ and $\#_b(w)$ differing.

NOW lets do a different kind of example:

$SQ = \{a^{n^2} : n \in \mathsf{N}\}$

Assume that L is regular. Assume the DFA is of of size s. Let n be large, we'll see how large later.

Let $w = a^{n^2}$. By the AGLP $w = xyz$ such that $|xy| \le 2s$ and $xy^*z \in L$.

Let $x = a^{n_1}$, $y = a^{n_2}$, $z = a^{n_3}$. We know that $n_2 \neq 0$ and

$n_1 + n_2 + n_3 = n^2$.

$w = a^{n_1} a^{n_2} a^{n_3}$

SO $a^{n_1} a^{2n_2} a^{n_3} = a^{n_1+2n_2+n_3}$.

Hence $n_1 + 2n_2 + n_3$ is a square.

Hence

$$n_1 + 2n_2 + n_3 \geq (n+1)^2 = n^2 + 2n + 1$$

$$(n_1 + n_2 + n_3) + n_2 \geq n^2 + 2n + 1$$

$$n^2 + n_2 \geq n^2 + 2n + 1$$

$$n_2 \geq 2n + 1$$

AH- but recall that $|xy| \leq 2s$ so $n_2 \leq 2s$.

Hence

$$2s \geq n_2 \geq 2n + 1$$

NOW we know how big to take n: take $n = s$.

This leads to the contradiction:

$$2s \geq 2s + 1$$