Homework 5 Due Mar 31 at 11:00AM (no dead cat extension)
THIS HOMEWORK IS TWO PAGES LONG!!!!!!!!!!!!!!!

1. (0 points, but if you actually miss the midterm without telling Dr. Gasarch ahead of time, you will lose 100 points on this homework) When will the midterm be (give date and time)? When will the final be (give date and time)? By when do you have to tell Dr. Gasarch that you cannot make the midterm?

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2. (100 points) For this problem you may use the following theorem.

**Theorem:** If $x, y$ are relatively prime then

- For all $z \geq xy - x - y + 1$ there exists $c, d \in \mathbb{N}$ such that $z = cx + dy$.
- There is no $c, d \in \mathbb{N}$ such that $xy - x - y = cx + dy$.

The alphabet is $\{a\}$. Let

$$L = \{a^n : n \neq 160\}$$

**WAITING:** Do not use the Pumping Lemma for this problem. We are NOT asking if $L$ is regular (it is!) we are asking about the number of states for a DFA for $L$ and an NFA for $L$.

(a) (30 points) Does there exist a DFA for $L$ with $\leq 159$ states? If so then draw the DFA; you may use DOT DOT DOT (You DO NOT have to prove that it works.) If not then PROVE there is no such DFA.

(b) (30 points) Does there exist a DFA for $L$ with $\leq 160$ states? If so then draw the DFA; you may use DOT DOT DOT (You DO NOT have to prove that it works.) If not then PROVE there is no such DFA.

(c) (40 points) Does there exist an NFA for $L$ with $\leq 50$ states? If so then draw the NFA; you may use DOT DOT DOT (You DO NOT have to prove that it works.) If not then PROVE there is no such NFA.
BEGIN SOLUTION

(a) No. Assume, by way of contradiction, that there was a DFA

\[ M = (Q, \Sigma, \delta, s, F) \]

where \(|Q| = 159\) that accepts \(L\). Feed \(a^{160}\) into \(M\) and let the sequence of states be

\[ q_0, q_1, q_2, \ldots, q_{160}. \]

Note that \(q_0 = s\) and \(q_{160} \notin F\).

Since \(|Q| = 159\) there exists \(i < j\) such that \(q_i = q_j\).

Hence we rewrite the state sequence as

\[ q_0, q_1, q_2, \ldots, q_i, q_{i+1}, \ldots, q_j, q_{j+1}, \ldots, q_{160} \]

We finish this proof one of two ways.

**WAY ONE:**

If you REMOVE the \(q_{i+1} \ldots q_j\) terms from the sequence then you have

\[ q_0, q_1, q_2, \ldots, q_i, q_{j+1}, \ldots, q_{160}. \]

Since \(q_i = q_j\), \(\delta(q_i, a) = q_{j+1}\). Hence the above state sequence is what happens when \(a^{160-j+i}\) is fed into \(M\) and it is rejected. Since

\[ 160 - j + i \neq 160 \]

this contradicts \(M\) accepting all but \(a^{160}\).

**WAY TWO:**

If you REPEAT the first \(q_{i+1}\) to \(q_{j-1}\) states then you have what happens when \(a^{160+j-i}\) is fed into \(M\) and it is rejected. Since \(160 + j - i \neq 160\) this contradicts \(M\) accepting all but \(a^{160}\).
(b) No. Assume, by way of contradiction, that there was a DFA

\[ M = (Q, \Sigma, \delta, s, F) \]

where \(|Q| = 160\). That accepts \(L\). Feed \(a^{160}\) into \(M\) let the sequence of states be

\[ q_0, q_1, q_2, \ldots, q_{160}. \]

Since \(|Q| = 160\) there is a repeat state. Then proof proceeds like part (a).

(c) Yes.
If \(x, y\) are rel prime then \(xy - x - y\) cannot be written as the sum of \(x\)'s and \(y\)'s, but any number larger can be.
We need to find \(x, y\) such that \(xy - x - y\) is just below 160.
\(x = 13\) and \(y = 14\). Then \(xy - x - y = 155\).
SO

- 155 CANNOT be written as \(13a + 14b\).
- Every number \(\geq 156\) can be written as \(13a + 14b\).
Hence

- 160 CANNOT be written as $5 + 13a + 14b$.
- Every number $\geq 161$ can be written as $5 + 14a + 15b$.

We now describe the NFA by describing its parts.

- From the start state there are 5 states in a line that end at state $q$. This takes 5 states. Make these 5 states FINAL states.
- Then we have a loop of size 14 with a one-edge 13 shortcut. This takes care of all $n \geq 161$. This takes 14 states.
- Also we have an $e$-transition to a DFA for $n \equiv 1 \mod 2$. This takes 2 states.
- Also we have an $e$-transition to a DFA for $n \equiv 0, 2 \mod 3$. This takes 3 states.
- Also we have an $e$-transition to a DFA for $n \equiv 1, 2, 3, 4 \mod 5$. This takes 5 states.
- Also we have an $e$-transition to a DFA for $n \equiv 0, 1, 2, 3, 4, 5 \mod 7$. This takes 7 states.

So the total number of states is

$$5 + 14 + 2 + 3 + 5 + 7 = 36.$$