Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

*If there are 6 people at a party, either 3 know each other or 3 do not know each other.*
Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

*If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

We state this in terms of colorings of edges of graphs.

*For all 2-coloring of the edges of $K_6$ there is a mono $K_3$.***
Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

*If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

We state this in terms of colorings of edges of graphs.

*For all 2-coloring of the edges of $K_6$ there is a mono $K_3$.*

**Question** What if we color the edges of $K_5$?
Coloring of $K_5$ with no Mono $K_3$

This graph is not arbitrary.

$SQ_5 = \{x^2 \pmod{5} : 0 \leq x \leq 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then RED.
- If $i - j \notin SQ_5$ then BLUE.
Asymmetric Ramsey Numbers

**Definition** $R(a, b)$ is least $n$ such that for all 2-colorings of $K_n$ there is either a red $K_a$ or a blue $K_b$.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$
Theorem \( R(a, b) \leq R(a - 1, b) + R(a, b - 1) \)

Proof

Let \( n = R(a - 1, b) + R(a, b - 1) \). COL: \( ([n]) \to [2] \).

Case 1 \((\exists v)[\deg_R(v) \geq R(a - 1, b)]\). Look at the \( R(a - 1, b) \) vertices that are RED to \( v \). By Definition of \( R(a - 1, b) \) either

- There is a RED \( K_{a-1} \). Combine with \( v \) to get RED \( K_a \).
- There is a BLUE \( K_b \).
Theorem \( R(a, b) \leq R(a - 1, b) + R(a, b - 1) \)

Proof
Let \( n = R(a - 1, b) + R(a, b - 1) \). COL: \( ([n]_2) \rightarrow [2] \).

Case 1 (\( \exists v \)[deg\(_R(v) \geq R(a - 1, b) \)]. Look at the \( R(a - 1, b) \) vertices that are RED to \( v \). By Definition of \( R(a - 1, b) \) either
  - There is a RED \( K_{a-1} \). Combine with \( v \) to get RED \( K_a \).
  - There is a BLUE \( K_b \).

Case 2 (\( \exists v \)[deg\(_B(v) \geq R(a, b - 1) \)]. Similar to Case 1.
Theorem \( R(a, b) \leq R(a - 1, b) + R(a, b - 1) \)

Proof
Let \( n = R(a - 1, b) + R(a, b - 1) \). COL: \( \left( \frac{n}{2} \right) \rightarrow [2] \).

Case 1 (\( \exists v \)[\( \deg_R(v) \geq R(a - 1, b) \)]). Look at the \( R(a - 1, b) \) vertices that are RED to \( v \). By Definition of \( R(a - 1, b) \) either

- There is a RED \( K_{a-1} \). Combine with \( v \) to get RED \( K_a \).
- There is a BLUE \( K_b \).

Case 2 (\( \exists v \)[\( \deg_B(v) \geq R(a, b - 1) \)]. Similar to Case 1.

Case 3
(\( \forall v \)[\( \deg_R(v) \leq R(a - 1, b) - 1 \land \deg_B(v) \leq R(a, b - 1) - 1 \)]
(\( \forall v \)[\( \deg(v) \leq R(a - 1, b) + R(a, b - 1) - 2 = n - 2 \)]
Not possible since every vertex of \( K_n \) has degree \( n - 1 \).
Lets Compute Bounds on $R(a, b)$

- $R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6$
- $R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 = 10$
- $R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + 10 = 15$
- $R(3, 6) \leq R(2, 6) + R(3, 5) \leq 6 + 15 = 21$
- $R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 21 = 28$
Lets Compute Bounds on $R(a, b)$

- $R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6$
- $R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 = 10$
- $R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + 10 = 15$
- $R(3, 6) \leq R(2, 6) + R(3, 5) \leq 6 + 15 = 21$
- $R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 21 = 28$

Can we make some improvements to this?
Let's Compute Bounds on $R(a, b)$

- $R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6$
- $R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 = 10$
- $R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + 10 = 15$
- $R(3, 6) \leq R(2, 6) + R(3, 5) \leq 6 + 15 = 21$
- $R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 21 = 28$

Can we make some improvements to this? YES!
$R(3, 4) \leq 9$

**Theorem** $R(3, 4) \leq 9$.

Let $COL$ be a 2-coloring of the edges of $K_9$.

**Case 1** $(\exists v)[\deg_R(v) \geq 4]$. $v_1, v_2, v_3, v_4$ are RED to $v$.
Theorem \( R(3, 4) \leq 9 \).

Let \( COL \) be a 2-coloring of the edges of \( K_9 \).

**Case 1** (\( \exists v \)[deg\(_R(v) \geq 4] \). \( v_1, v_2, v_3, v_4 \) are RED to \( v \).
If any of \( v_i, v_j \) is RED, then \( v, v_i, v_j \) are RED \( K_3 \).
Theorem $R(3, 4) \leq 9$.

Let $COL$ be a 2-coloring of the edges of $K_9$.

**Case 1** $(\exists v)[\deg_R(v) \geq 4]$. $v_1, v_2, v_3, v_4$ are RED to $v$.
If any of $v_i, v_j$ is RED, then $v, v_i, v_j$ are RED $K_3$.  
If not then $v_1, v_2, v_3, v_4$ is BLUE $K_4$. 

**Case 2** $(\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are BLUE to $v$.
Either:
(1) a RED $K_3$, or
(2) a BLUE $K_3$, which together with $v$ is a BLUE $K_4$.

**Case 3** $(\forall v)[\deg_R(v) = 3]$. The RED subgraph has 9 nodes each of degree 3. Impossible!
Theorem $R(3, 4) \leq 9$.

Let $COL$ be a 2-coloring of the edges of $K_9$.

**Case 1** ($\exists v)[\deg_R(v) \geq 4]$. $v_1, v_2, v_3, v_4$ are RED to $v$.
If any of $v_i, v_j$ is RED, then $v, v_i, v_j$ are RED $K_3$.
If not then $v_1, v_2, v_3, v_4$ is BLUE $K_4$.

**Case 2** ($\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are BLUE to $v$.
Either:
(1) a RED $K_3$, or
(2) a BLUE $K_3$, which together with $v$ is a BLUE $K_4$.

**NOTE** Can’t have any $\deg_R(v) \leq 2$. 
Theorem \( R(3, 4) \leq 9 \).
Let COL be a 2-coloring of the edges of \( K_9 \).

**Case 1** (\( \exists v \)[\( \deg_R(v) \geq 4 \)]. \( v_1, v_2, v_3, v_4 \) are RED to \( v \).
If any of \( v_i, v_j \) is RED, then \( v, v_i, v_j \) are RED \( K_3 \).
If not then \( v_1, v_2, v_3, v_4 \) is BLUE \( K_4 \).

**Case 2** (\( \exists v \)[\( \deg_B(v) \geq 6 \)]. \( v_1, v_2, v_3, v_4, v_5, v_6 \) are BLUE to \( v \).
Either:
(1) a RED \( K_3 \), or
(2) a BLUE \( K_3 \), which together with \( v \) is a BLUE \( K_4 \).

**NOTE** Can’t have any \( \deg_R(v) \leq 2 \).

**Case 3** (\( \forall v \)[\( \deg_R(v) = 3 \)]. The RED subgraph has 9 nodes each of degree 3. Impossible!
Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{ v : \deg(v) \equiv 0 \pmod{2} \}$$
$$V_{\text{odd}} = \{ v : \deg(v) \equiv 1 \pmod{2} \}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod{2}$. 
Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{v : \deg(v) \equiv 0 \pmod{2}\}$$
$$V_{\text{odd}} = \{v : \deg(v) \equiv 1 \pmod{2}\}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod{2}$.

Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod{2}.$$
Lemma Let $G = (V, E)$ be a graph.

$$V_{\text{even}} = \{ v : \deg(v) \equiv 0 \pmod{2} \}$$
$$V_{\text{odd}} = \{ v : \deg(v) \equiv 1 \pmod{2} \}$$

Then $|V_{\text{odd}}| \equiv 0 \pmod{2}$.

Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod{2}.$$
Lemma Let $G = (V, E)$ be a graph.

\[ V_{\text{even}} = \{ v : \deg(v) \equiv 0 \pmod{2} \} \]
\[ V_{\text{odd}} = \{ v : \deg(v) \equiv 1 \pmod{2} \} \]

Then $|V_{\text{odd}}| \equiv 0 \pmod{2}$.

Recall that for any graph $G = (V, E)$:

\[ \sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod{2}. \]

\[ \sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod{2}. \]

Sum of odds $\equiv 0 \pmod{2}$. Must have even numb of them. So $|V_{\text{odd}}| \equiv 0 \pmod{2}$. 
A Generalization of this Trick

What was it about $R(3, 4)$ that made that trick work?
A Generalization of this Trick

What was it about $R(3, 4)$ that made that trick work? We originally had

$$R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 \leq 10$$
A Generalization of this Trick

What was it about $R(3, 4)$ that made that trick work?
We originally had

$$R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 \leq 10$$

**Key:** $R(2, 4)$ and $R(3, 3)$ were both even!
A Generalization of this Trick

What was it about $R(3, 4)$ that made that trick work?
We originally had

$$R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 \leq 10$$

**Key:** $R(2, 4)$ and $R(3, 3)$ were both *even!*

**Theorem** $R(a, b) \leq$

1. $R(a, b - 1) + R(a - 1, b)$ always.
2. $R(a, b - 1) + R(a - 1, b) - 1$ if
   $R(a, b - 1) \equiv R(a - 1, b) \equiv 0 \pmod{2}$
Some Better Upper Bounds

$R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6.$

$R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 - 1 = 9.$

$R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + 9 = 14.$

$R(3, 6) \leq R(2, 6) + R(3, 5) \leq 6 + 14 - 1 = 19.$

$R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 19 = 26$

$R(4, 4) \leq R(3, 4) + R(4, 3) \leq 9 + 9 = 18.$

$R(4, 5) \leq R(3, 5) + R(4, 4) \leq 14 + 18 - 1 = 31.$

$R(5, 5) \leq R(4, 5) + R(5, 4) = 62.$

Are these tight?
\[ R(3, 3) \geq 6 \]

\[ R(3, 3) \geq 6: \text{ Need coloring of } K_5 \text{ w/o mono } K_3. \]
\[ R(3, 3) \geq 6 \]

\[ R(3, 3) \geq 6: \text{ Need coloring of } K_5 \text{ w/o mono } K_3. \]

Vertices are \( \{0, 1, 2, 3, 4\} \).
$R(3, 3) \geq 6$

$R(3, 3) \geq 6$: Need coloring of $K_5$ w/o mono $K_3$.

Vertices are $\{0, 1, 2, 3, 4\}$.

$COL(a, b) =$ RED if $a - b \equiv SQ \pmod{5}$, BLUE OW.
\( R(3,3) \geq 6 \)

\( R(3,3) \geq 6 \): Need coloring of \( K_5 \) w/o mono \( K_3 \).

Vertices are \( \{0, 1, 2, 3, 4\} \).

\( \text{COL}(a, b) = \text{RED} \) if \( a - b \equiv SQ \pmod{5} \), \( \text{BLUE} \) OW.

**Note** \(-1 = 2^2 \pmod{5}\). Hence \( a - b \in \text{SQ} \) iff \( b - a \in \text{SQ} \). So the coloring is well defined.
\[ R(3, 3) \geq 6 \]

\[
\text{COL}(a, b) = \text{RED if } a - b \equiv SQ \pmod{5}, \text{ BLUE OW.}
\]

- Squares mod 5: 1, 4.
- If there is a RED triangle then \(a - b, b - c, c - a\) all SQ’s. SUM is 0. So

\[
x^2 + y^2 + z^2 \equiv 0 \pmod{5} \text{ Can show impossible}
\]

- If there is a BLUE triangle then \(a - b, b - c, c - a\) all non-SQ’s. Product of nonsq’s is a sq. So
  \(2(a - b), 2(b - c), 2(c - a)\) all squares. SUM to zero- same proof.

**UPSHOT** \( R(3, 3) = 6 \) and the coloring used math of interest!
$R(4, 4) = 18$

$R(4, 4) \geq 18$: Need coloring of $K_{17}$ w/o mono $K_4$. 
$R(4, 4) = 18$

$R(4, 4) \geq 18$: Need coloring of $K_{17}$ w/o mono $K_4$.

Vertices are $\{0, \ldots , 16\}$.

Use

$COL(a, b) = \text{RED}$ if $a - b \equiv SQ \pmod{17}$, BLUE OW.
\( R(4, 4) = 18 \)

\( R(4, 4) \geq 18 \): Need coloring of \( K_{17} \) w/o mono \( K_4 \).

Vertices are \( \{0, \ldots, 16\} \).

Use

\[
\text{COL}(a, b) = \text{RED} \text{ if } a - b \equiv SQ \pmod{17}, \text{ BLUE OW.}
\]

Same idea as above for \( K_5 \), but more cases.

**UPSHOT** \( R(4, 4) = 18 \) and the coloring used math of interest!
$R(3, 5) = 14$

$R(3, 5) \geq 14$: Need coloring of $K_{13}$ w/o RED $K_3$ or BLUE $K_5$. 
$R(3, 5) = 14$

$R(3, 5) \geq 14$: Need coloring of $K_{13}$ w/o RED $K_3$ or BLUE $K_5$.

Vertices are $\{0, \ldots, 13\}$.

Use

$COL(a, b) = \text{RED}$ if $a - b \equiv \text{CUBE} \pmod{14}$, BLUE OW.
\[ R(3, 5) = 14 \]

\[ R(3, 5) \geq 14: \text{ Need coloring of } K_{13} \text{ w/o RED } K_3 \text{ or BLUE } K_5. \]

Vertices are \{0, \ldots, 13\}.

Use \[ \text{COL}(a, b) = \text{RED if } a - b \equiv CUBE \pmod{14}, \text{ BLUE OW}. \]

Same idea as above for \( K_5 \), but more cases.
$R(3, 5) = 14$

$R(3, 5) \geq 14$: Need coloring of $K_{13}$ w/o RED $K_3$ or BLUE $K_5$.

Vertices are $\{0, \ldots, 13\}$.

Use

$$COL(a, b) = \text{RED if } a - b \equiv \text{CUBE} \Mod{14}, \text{ BLUE OW.}$$

Same idea as above for $K_5$, but more cases.

**UPSHOT** $R(3, 5) = 14$ and the coloring used math of interest!
$R(3, 4) = 9$

This is a subgraph of the $R(3, 5)$ graph
$R(3, 4) = 9$

This is a subgraph of the $R(3, 5)$ graph

**UPSHOT**  $R(3, 4) = 9$ and the coloring used math of interest!
Can we extend these Patterns?

**Good news** $R(4, 5) = 25.$
Can we extend these Patterns?

**Good news** $R(4, 5) = 25$.

**Bad news**
Can we extend these Patterns?

**Good news** \( R(4, 5) = 25. \)

**Bad news**
THATS IT.
Can we extend these Patterns?

**Good news** \( R(4, 5) = 25. \)

**Bad news**

THATS IT.

No other \( R(a, b) \) are known using NICE methods.
Can we extend these Patterns?

**Good news** $R(4, 5) = 25$.

**Bad news**

THATS IT.

No other $R(a, b)$ are known using NICE methods.

$R(5, 5)$ – I will give you a paper to read on that soon.
Revisit those Numbers


- $R(3, 3) \leq 6$. TIGHT. Int
- $R(3, 4) \leq 9$. TIGHT. Int
- $R(3, 5) \leq 14$. TIGHT. Int
- $R(3, 6) \leq 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3, 7) \leq 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4, 4) \leq 18$. TIGHT. Int
- $R(4, 5) \leq 31$. KNOWN: 25. Both bd Bor
- $R(5, 5) \leq 62$. KNOWN: Will see it in the paper I give out.
Moral of the Story

1. At first there seemed to be interesting mathematics with mods and primes leading to nice graphs.
Moral of the Story

1. At first there seemed to be interesting mathematics with mods and primes leading to nice graphs. (Joel Spencer) *The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.*
Moral of the Story

1. At first there seemed to be interesting mathematics with mods and primes leading to nice graphs. (Joel Spencer) *The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.*

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.
1. (Quote from Joel Spencer): Erdos asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.
1. (Quote from Joel Spencer): *Erdos asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens.*

2. I asked Stanislaw Radziszowski, the worlds leading authority on Small Ramsey Numbers, what R(5, 5) is and when we would know it. He said its 43 and we will never know it.
Upper Bounds on $R(k)$

Recall that $R(a, 2) = a$,

$R(2, b) = b$,

$R(a, b) \leq R(a, b-1) + R(a-1, b)$.

We use these to get an upper bound on $R(k) = R(k, k)$. Discuss!
Upper Bounds on $R(k)$

Recall that

$$R(a, 2) = a$$
Upper Bounds on $R(k)$

Recall that

$R(a, 2) = a$

$R(2, b) = b$
Upper Bounds on $R(k)$

Recall that 

$R(a, 2) = a$

$R(2, b) = b$

$R(a, b) \leq R(a, b - 1) + R(a - 1, b)$
Upper Bounds on $R(k)$

Recall that

$R(a, 2) = a$

$R(2, b) = b$

$R(a, b) \leq R(a, b - 1) + R(a - 1, b)$

We use these to get an upper bound on $R(k) = R(k, k)$. 

Discuss!
Upper Bounds on $R(k)$

Recall that

$R(a, 2) = a$

$R(2, b) = b$

$R(a, b) \leq R(a, b - 1) + R(a - 1, b)$

We use these to get an upper bound on $R(k) = R(k, k)$. Discuss!
Upper Bounds on $R(k)$ (cont)

**Thm** For all $a, b \geq 2$, $R(a, b) \leq 2^{a+b}$.
Thm For all $a, b \geq 2$, $R(a, b) \leq 2^{a+b}$.

Proof by Induction on $a + b$
Upper Bounds on $R(k)$ (cont)

**Thm** For all $a, b \geq 2$, $R(a, b) \leq 2^{a+b}$.

**Proof by Induction on $a + b$**

**Base** $a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq 2^{2+2} = 2^4$. 
Upper Bounds on $R(k)$ (cont)

**Thm** For all $a, b \geq 2$, $R(a, b) \leq 2^{a+b}$.

**Proof by Induction on** $a + b$

**Base** $a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq 2^{2+2} = 2^4$.

**Ind Hyp** $(\forall a', b', a' + b' < a + b)[R(a', b') \leq 2^{a'+b'}]$. 
Upper Bounds on $R(k)$ (cont)

**Thm** For all $a, b \geq 2$, $R(a, b) \leq 2^{a+b}$.

**Proof by Induction on $a + b$**

**Base** $a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq 2^{2+2} = 2^4$.

**Ind Hyp** $(\forall a', b', a' + b' < a + b)[R(a', b') \leq 2^{a'+b'}]$.

**Ind Step**

$$R(a, b) \leq R(a, b-1) + R(a-1, b) \leq 2^{a+b-1} + 2^{a-1+b} = 2 \times 2^{a+b-1} = 2^{a+b}.$$

**End of Proof**

**Corollary** $R(k) = R(k, k) \leq 2^{k+k} = 2^{2k}$. 
Can We Do Better?

Vote
Can We Do Better?

Vote

1. \( R(k) \leq 2^{2k} / \sqrt{k} \).
Can We Do Better?

Vote

1. $R(k) \leq 2^{2k}/\sqrt{k}$.
2. $R(k) \leq 2^k$. 

Answer on next slide.
Can We Do Better?

Vote

1. \( R(k) \leq 2^{2k/\sqrt{k}}. \)
2. \( R(k) \leq 2^k. \)
3. \( R(k) \leq 2^{k/2} \)
Can We Do Better?

Vote

1. $R(k) \leq 2^{2k}/\sqrt{k}$.
2. $R(k) \leq 2^k$.
3. $R(k) \leq 2^{k/2}$
4. $R(k) \leq 2^{k/3}$
Can We Do Better?

Vote

1. \( R(k) \leq 2^{2k}/\sqrt{k} \).
2. \( R(k) \leq 2^k \).
3. \( R(k) \leq 2^{k/2} \).
4. \( R(k) \leq 2^{k/3} \).
5. \( R(k) \leq k^{1000} \).
Vote

1. \( R(k) \leq 2^{2k} / \sqrt{k}. \)
2. \( R(k) \leq 2^k. \)
3. \( R(k) \leq 2^{k/2} \)
4. \( R(k) \leq 2^{k/3} \)
5. \( R(k) \leq k^{1000}. \)

Answer on next slide.
What We Know

Known

1. \( R(k) \leq 2^{k/\sqrt{k}}. \) TRUE. We will do this.

2. \( R(k) \leq 2^k. \) UNKNOWN TO SCIENCE.

3. \( R(k) \leq 2^{k/2}. \) FALSE by just a little. We will show this.

4. \( R(k) \leq 2^{k/3}. \) FALSE.

5. \( R(k) \leq 1000^{1000}. \) FALSE.
What We Know

Known

1. $R(k) \leq 2^{2^k} / \sqrt{k}$. 
What We Know

Known

1. $R(k) \leq \frac{2^{2k}}{\sqrt{k}}$. TRUE. We will do this.
What We Know

Known

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$.
What We Know

**Known**

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
What We Know

**Known**

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$.
What We Know

**Known**

1. $R(k) \leq 2^{2k} / \sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$. FALSE by just a little. We will show this.
What We Know

**Known**

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$. FALSE by just a little. We will show this.
4. $R(k) \leq 2^{k/3}$.
What We Know

**Known**

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$. FALSE by just a little. We will show this.
4. $R(k) \leq 2^{k/3}$. FALSE.
What We Know

**Known**

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$. FALSE by just a little. We will show this.
4. $R(k) \leq 2^{k/3}$. FALSE.
5. $R(k) \leq k^{1000}$. FALSE.
What We Know

Known

1. $R(k) \leq 2^{2k}/\sqrt{k}$. TRUE. We will do this.
2. $R(k) \leq 2^k$. UNKNOWN TO SCIENCE.
3. $R(k) \leq 2^{k/2}$. FALSE by just a little. We will show this.
4. $R(k) \leq 2^{k/3}$. FALSE.
5. $R(k) \leq k^{1000}$. FALSE.
Lemma Needed For Better Bounds on $R(k)$

Thm $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. 

Proof: We could prove by algebra. We are too cool for that! $\binom{n}{k}$ is the number of ways of choosing $k$ numbers out of $\{1, \ldots, n\}$.

We split this problem into two problems. $n$ is not chosen. Then there are $\binom{n-1}{k}$ ways to do to. $n$ is chosen. Then there are $\binom{n-1}{k-1}$ ways to do to. So the number of ways to choose $k$ numbers out of $\{1, \ldots, n\}$ is $\binom{n-1}{k} + \binom{n-1}{k-1}$. 
Lemma Needed For Better Bounds on $R(k)$

Thm $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof We could prove by algebra.
Lemma Needed For Better Bounds on $R(k)$

$$\text{Thm } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$  

**Proof** We could prove by algebra. We are too cool for that!
Lemma Needed For Better Bounds on $R(k)$

**Thm** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

**Proof** We could prove by algebra. We are too cool for that!

$\binom{n}{k}$ is the number of ways of choosing $k$ numbers out of $\{1, \ldots, n\}$. 
Lemma Needed For Better Bounds on $R(k)$

**Thm** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

**Proof** We could prove by algebra. We are too cool for that!

$\binom{n}{k}$ is the number of ways of choosing $k$ numbers out of $\{1, \ldots, n\}$. We split this problem into two problems.
Lemma Needed For Better Bounds on $R(k)$

**Thm** \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \).

**Proof** We could prove by algebra. We are too cool for that!

\( \binom{n}{k} \) is the number of ways of choosing \( k \) numbers out of \( \{1, \ldots, n\} \).

We split this problem into two problems.

\( n \) is not chosen. Then there are \( \binom{n-1}{k} \) ways to do to.

\( n \) is chosen. Then there are \( \binom{n-1}{k-1} \) ways to do to.
Lemma Needed For Better Bounds on $R(k)$

**Thm** \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \).

**Proof** We could prove by algebra. We are too cool for that!

\( \binom{n}{k} \) is the number of ways of choosing \( k \) numbers out of \( \{1, \ldots, n\} \).

We split this problem into two problems.

- **n is not** chosen. Then there are \( \binom{n-1}{k} \) ways to do to.
- **n is** chosen. Then there are \( \binom{n-1}{k-1} \) ways to do to.
Lemma Needed For Better Bounds on $R(k)$

**Thm** \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \).

**Proof** We could prove by algebra. We are too cool for that!

\( \binom{n}{k} \) is the number of ways of choosing \( k \) numbers out of \{1, \ldots, n\}. We split this problem into two problems.

- **n is not** chosen. Then there are \( \binom{n-1}{k} \) ways to do to.
- **n is** chosen. Then there are \( \binom{n-1}{k-1} \) ways to do to.

So the number of ways to choose \( k \) numbers out of \{1, \ldots, n\} is
Lemma Needed For Better Bounds on $R(k)$

**Thm** $(\binom{n}{k}) = (\binom{n-1}{k}) + (\binom{n-1}{k-1})$.

**Proof** We could prove by algebra. We are too cool for that!

$(\binom{n}{k})$ is the number of ways of choosing $k$ numbers out of $\{1, \ldots, n\}$. We split this problem into two problems.

$n$ is not chosen. Then there are $(\binom{n-1}{k})$ ways to do so.

$n$ is chosen. Then there are $(\binom{n-1}{k-1})$ ways to do so.

So the number of ways to choose $k$ numbers out of $\{1, \ldots, n\}$ is

$$\binom{n-1}{k} + \binom{n-1}{k-1}.$$
Better Bounds on $R(k)$

**Thm** $R(a, b) \leq \binom{a+b-2}{a-1}$.
Better Bounds on $R(k)$

**Thm** $R(a, b) \leq \binom{a+b-2}{a-1}$. **Proof by Induction on** $a + b$
**Better Bounds on** $R(k)$

**Thm** $R(a, b) \leq \binom{a+b-2}{a-1}$. **Proof by Induction on** $a + b$ **Base**

$a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq \binom{2}{1} = 2$.
Better Bounds on $R(k)$

**Thm** $R(a, b) \leq \binom{a+b-2}{a-1}$. **Proof by Induction on** $a + b$ Base $a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq \binom{2}{1} = 2$.

**IH** $(\forall a', b', a' + b' < a + b)[R(a', b') \leq \binom{a'+b'-2}{a'-1}]$. 

**IS** $R(a, b) \leq R(a, b-1) + R(a-1, b) \leq \binom{a+b-3}{a-1} + \binom{a+b-2}{a-2} = \binom{a+b-2}{a-1}$ by the Lemma.

**End of Proof**

**Cor** $R(k) = R(k, k) \leq \binom{2k-3}{k-1} \leq 2^{\sqrt{k}}$. 
Better Bounds on $R(k)$

**Thm** $R(a, b) \leq \binom{a+b-2}{a-1}$. **Proof by Induction on** $a + b$ **Base**

$a + b = 4$ so $a = b = 2$. $R(2, 2) = 2 \leq \binom{2}{1} = 2.$

**IH** $(\forall a', b', a' + b' < a + b) [R(a', b') \leq \binom{a' + b' - 2}{a' - 1}].$

**IS**

\[
R(a, b) \leq R(a, b - 1) + R(a - 1, b) \leq \binom{a + b - 3}{a - 2} + \binom{a + b - 3}{a - 1}
\]

\[
= \binom{a + b - 2}{a - 1} \text{ by the Lemma.}
\]

**End of Proof**

**Cor** $R(k) = R(k, k) \leq \binom{2k - 2}{k - 1} \leq 2^{2k} / \sqrt{k}.$
Slightly Better Upper Bounds are Known

In 2006 David Conlon showed that there is a $C$ such that

$$R(k) \leq k - C \log k \log \log k (2k^k).$$
Slightly Better Upper Bounds are Known

In 2006 David Conlon showed that there is a $C$ such that

$$R(k) \leq k^{-C \frac{\log k}{\log \log k}} \binom{2k}{k}.$$
In 2006 David Conlon showed that there is a $C$ such that

$$R(k) \leq k^{-C \frac{\log k}{\log \log k}} \binom{2k}{k}.$$ 

The proof is difficult and we won’t be doing it.

The link is [https://arxiv.org/abs/math/0607788](https://arxiv.org/abs/math/0607788)
In 2006 David Conlon showed that there is a $C$ such that

$$R(k) \leq k^{-C \frac{\log k}{\log \log k}} \binom{2k}{k}.$$ 

The proof is difficult and we won't be doing it.

The link is https://arxiv.org/abs/math/0607788