When Is There R^2 with no Red L_3 or Blue $L_{\rm big}$
Exposition by William Gasarch, Chaewoon Kyoung, Kelin Zhu

1 Pre Introduction

The following is well known.

Theorem 1.1 *For all* COL: $\mathsf{R}^2 \rightarrow [2]$ *there exists 2 points, same color, 1 inch apart.*

We rephrase this but first need some definitions.

Definition 1.2

1. ℓ_2 is 2 points in the plane an inch apart.
2. ℓ_3 is three colinear points p_1, p_2, p_3 where $d(p_1, p_2) = d(p_2, p_3) = 1$.
3. You can define ℓ_k.
4. Given COL: $\mathsf{R}^2 \rightarrow [2]$, a *RED* ℓ_k is an ℓ_k where all the points in it are RED. Similar for a BLUE ℓ_k.

In this paper we present proves of the following two known Theorems.

1. (Szlam [5]) There exists a constant c' such that $\mathsf{R}^n \rightarrow (\ell_2, \ell_m)$ where $m = 2^{c'n}$. (He proved a more general theorem. See his paper for details.)
2. (Conlon & Fox [1]) There exists a constant d' such that $\mathsf{R}^n \not\rightarrow (\ell_2, \ell_m)$ where $m = 2^{d'n}$. We will just prove the $n = 2$ case in this paper. (They proved a more general theorem. See their paper for details.)

BILL- WE MAY ADD THE GENERAL CASE LATER

2 Lemma Needed To Show $\mathsf{R}^n \rightarrow (\ell_2, \ell_m)$ where $m = 2^{c'n}$

Notation 2.1 Let $G_n = (V, E)$ be the graph with $V = \mathsf{R}^n$ and $E = \{(x, y) : d(x, y) = 1\}$. Let $c(n)$ be the chromatic number of G_n.

It is well known that $5 \le c(2) \le 7$.
The following is known:

Theorem 2.2

1. *(Larman and Rogers [3])* $c(n) \le (3 + o(1))^n$
2. *(Raigorodskii [4])* $c(n) \le (1.239\ldots + o(1))^n$
3. *(Frankl and Wilson [2])* $c(n) \ge (1 + o(1))(1.2)^n$. *We use the following easier-to-use version: There exists* c' *such that* $c(n) \le 2^{c'n}$.

BILL: WILL LATER FILL IN PROOFS OF ALL THREE OF THESE.

3 $\mathsf{R}^n \rightarrow (\ell_2, \ell_m)$ where $m = 2^{c'n}$

Theorem 3.1 *(Szlam [5]) There exists* c' *such that* $\mathsf{R}^n \rightarrow (\ell_2, \ell_m)$ *where* $m = 2^{c'n}$.

Proof: We will need the following notation: $\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in R^n.

Let COL: $\mathsf{R}^n \rightarrow [2]$.

Case 1 There is a BLUE ℓ_m. Done

Case 2 There is no BLUE ℓ_m. We form a coloring COL: $\mathsf{R}^n \rightarrow [m]$ as follows:

Given point $p \in \mathsf{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1}) = \mathrm{RED}$. Color p with the least such i.

By Theorem 2.2 there exists points $p, q \in \mathsf{R}^n$ and $1 \le i \le m$ such that $d(p, q) = 1$ and p, q are the same color. Hence $p + i\vec{1}$ and $q + i\vec{1}$ are both RED. Since $d(p, q) = 1$, $d(p + i\vec{1}, q + i\vec{1}) = 1$. Hence $p + i\vec{1}$ and $q + i\vec{1}$ form a RED ℓ_2. ∎

4 Lemmas Needed To Show $\mathsf{R}^n \not\rightarrow (\ell_2, \ell_m)$ where $m = 2^{d'n}$

We will be 2-coloring the $m \times m$ square and then use that to form a periodic coloring of R^2. Hence we think of coloring the $m \times m$ with the two horizontal sides identified and the new vertical sides identified. We denote this T_m^2 (The T is for Taurus.)

BILL- THE PAPER USES $m \times m$. I WILL LATER SAY WHY I USE $m \times m$.

BILL: WE NEED A PICTURE FOR AN EXAMPLE. KELIN CAN DO THIS WITH A COLOR PICTURE OF A SQUARE, LIKE HE DID A COLOR BULLSEYE IN THE $L6$-$L6$ PAPER.

We need several lemmas.

Definition 4.1 Let $t \in \mathsf{R}^+$. Let $P \subseteq T_m^2$.

1. P is t-separated if, for all $p, q \in P$, $d(p, q) \ge t$.
2. P is *maximally t-separated* (1) if P is t-separated and (2) for all $r \notin P$, $P \cup \{r\}$ is not t-separated.

Lemma 4.2 *Let* $t \in \mathsf{R}^+$ *and* $m \in \mathsf{N}$.

1. *There exists* $P \subseteq T_m^2$ *that is maximally t-seperated.*
2. *If* $P \subseteq T_m^2$ *is maximally t-seperated then* $|P| \le \frac{(m/t)^2}{\pi}$.
3. *If* $P \subseteq T_m^2$ *is maximally* $\frac{1}{3}$*-seperated then* $|P| \le (1.7m)^2$. *This follows from Part 2.*

Proof:
1) A greedy algorithm forms a maximally t-seperated set.
BILL: How fast is this? Can we get a faster algorithm?
2) Let $p \in P$. Then there is no element of P inside the circle centered at p of radius t. This circle has area πt^2. The set T_m^2 has area m^2. Hence
$|P| \times \pi t^2 \leq m^2$, so $|P| \leq \frac{(m/t)^2}{\pi}$. ∎

Lemma 4.3 *Let $t \in \mathsf{R}^+$. Let $S \subseteq \mathsf{R}^2$ be t-seperated. Let $\vec{p} \in \mathsf{R}^2$. Let $s \geq 0$. The number of points of S within s of $\vec{p}$ is at most $(2s/t+1)^2$.*

Proof: Let T be the set of points within t of $\vec{p}$. For every $\vec{q} \in T$ we look at the circle centered at $\vec{q}$ of radius $t/2$ (we can't use radius t since then the circles would not be disjoint). These circles have no other points of T in them and are disjoint. These circles have area $\pi(t/2)^2$. The union of these circles is contained in the circle around $\vec{p}$ of radius $s + t/2$ which has area $\pi(s+t/2)^2$. Hence
$|T| \times \pi t^2/4 \leq \pi(s+t/2)^2$
$|T| \times (t/2)^2 \leq (s+t/2)^2$
$|T| \leq (\frac{s+t/2}{t/2})^2 = (2s/t+1)^2$. ∎

Definition 4.4 Assume $S \subseteq \mathsf{R}^2$ or $S \subseteq T_2^m$. If $p \in S$ then V_p is the set of points of R^2 or T_2^m that are closer (or tied) to p then to any other point of S. The *Voronoi Diagram of S* is the set of all the V_p's.

BILL- DO EXAMPLES

1. A NORMAL EXAMPLE

2. AN EXAMPLE WHERE THE VORONOI CELL IS A POLYGON WITH LOTS OF SIDES. I THINK IF THE SET OF POINTS IS A p AND m POINTS ON THE CIRCLE OF RADIUS 1 AROUND x THEN V_p would be a m-sided convex polygon.

Note 4.5 There exists $S \subseteq \mathsf{R}^n$ and an $s \in S$ such that V_p is a convex $|S|$-gon. See BILL-WILL NEED FIGURE NUMBER.

Lemma 4.6 *Let $S \subseteq \mathsf{R}^2$ be a maximal t-separated set. We form the Voronoi diagram of S. The Voronoi cells are $\{V_p\}_{p \in S}$.*

1. *If $x \in V_p$ then $d(x,p) \leq t$.*

2. *If $p, p' \in V_p$ then $d(p,p') \leq 2t$. (This follows from Part 1.)*

3. *If $p, p' \in S$ and V_p, $V_{p;}$ share a boundary then $d(p,p') \leq 2t$.*

4. *V_p is convex polygon with ≤ 25 sides.*

Proof:
1) Assume, by way of contradiction, that there is an $x \in V_p$ and $d(x,p) > t$. Since $x \in V_p$, $d(x,p)$ is the smallest distance from x to a point of S. Hence x is greater than t away from any point in S. Since S is maximal, $x \in S$ which is a contradiction.
3) Draw a line from p to p'. It will hit a point x that is on both the boundary of V_p and the boundary of $V_{p'}$. By Part 1
$d(p,p') = d(p,x) + d(x,p') \le t + t = 2t$.
4) V_p is a convex polygon. Map each side of V_p to the p' such that V_p and $V_{p'}$ share that side. Using Part 2 we get that the number of sides is bounded above by the number of points of $p' \in S$ such that $d(p,p') \le 2t$. By Lemma 4.3 the number of such points is $\le ((2 \times 2t)/t + 1)^2 = 5^2 = 25$. ∎

BILL- I DO NOT THINK I NEED THE LEMMA BELOW FOR THE THEOREM. THEY NEED TO USE A SET OF SIZE $m/5$ THAT HAS POINTS 5 APART. WE WILL JUST NEED THAT ℓ_m DOES NOT HIT TWO ANALOGOUS VORONOI CELLS FROM DIFF TILES. THIS WILL BE ACCOMPLISHED BY MAKING THE TILES $m \times m$ SINCE THE MAX DISTANCE BETWEEN POINTS OF ℓ_m IS $m-1$. THE PAPER DOES MORE COMPLICATED THINGS

Lemma 4.7 *Let K be a 1-seperated set. Let $s \ge 1$. There is a set $K' \subseteq K$ that is s-separated such that $|K'| \ge |K|/(2s+1)^2$.*

5 $\mathsf{R}^n \not\to (\ell_2, \ell_m)$ where $m = 2^{d'n}$

Theorem 5.1 *There exists d' such that $\mathsf{R}^2 \not\to (\ell_2, \ell_m)$ where $m = 2^{d'n}$.*

Proof: Let P be a maximal $\frac{1}{3}$-separated subset of T_2^m. We create the Voronoi diagram of P.

Let $Q \subseteq P$ be formed by, for each $p \in P$, choose it with probability x (we will determine x later).

Let $S \subseteq Q$ be the set of points $s \in Q$ such that, for all $s' \in Q$, $d(s,s') > 5/3$.

Recall that we have a Voronoi diagram formed by the points in P. Let the Voronoi cells that have a point of S in them be denoted $V_1, \ldots, V_{|S|}$.

We will color each V_i, including boundary, RED. We will color every other point in T_2^m BLUE. We will then use this to periodically color R^2. We view this as tiling the plane with $m \times m$ tiles and coloring all the tiles the same.

We will show that if you take a nine tiles arrange 3×3 then there is no RED ℓ_2 or BLUE ℓ_m with a point in the middle tile. This will suffice.

No RED ℓ_2 This part does not use probability.

Let q, q' both be RED.

Case 1: q, q' are in the same Voronoi cell. By Lemma 4.6.2 $d(q,q') \le 1/3$.

Case 2: q, q' are in the same tile but in different Voronoi cells. Let the Voronoi cells have centers p, p'. Then

$$d(p,p') \le d(p,q) + d(q,q') + d(q',p') \le \frac{1}{3} + 1 + \frac{1}{3} = \frac{5}{3}.$$

But by definition of S, $d(p,p') > \frac{5}{3}$.

Case 3: q, q' are in different tiles but in the analogous Voronoi cells. Let the Voronoi cells have centers p, p'. Since $d(p,p') = m$, $d(q,q') \ge m - \frac{1}{3} > 1$.

Case 4: q, q' are in different tiles and non-analogous Voronoi cells. Since the Voronoi diagram was on a Taurus this is identical to Case 2.

No BLUE ℓ_m

Let $L = (q_1, \ldots, q_m)$ be an ℓ_m. We bound the probability that L is BLUE.

Let $\{p_i\}_{i=0}^{m'-1}$ be such that, for $0 \le i \le m' - 1$, $q_i \in V_{p_i}$. We need to bound the probability that V_{p_i} is BLUE. Not so fast! We need to show that all of the V_{p_i} are distinct.

Let $q, q' \in \{q_0, \ldots, q_{m'-1}\}$. Let $\{p, p'\}$ be such that $q \in V_p$ and $q' \in V_{p'}$.

Case 1 q, q' are in the same tile and in the same Voronoi cell. This cannot happen since $d(q, q') \ge 1$ and by Lemma 4.6.2 the diameter of these cells is $2/3$.

Case 2 q, q' are in the different tiles but in analogous Voronoi cells. Two points in analogous cells are at least $m - \frac{2}{3}$ apart. Since $d(q, q') \le m - 1$, q, q' cannot be in different tiles but in analogous Voronoi cells.

The probability that L is BLUE is the prob that V_{p_1}, V_{p_2}, …, V_{p_m} are all BLUE.

Let $p \in P$. We determine a lower bound on the probability that V_p is RED. Recall that V_p is RED iff $p \in S$.

∎

References

[1] D. Conlon and J. Fox. Line in Euclidean Ramsey theory. *Discrete and Computational Geometry*, 5:218–225, 2017.

[2] P. Frankl and R. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1:357–368, 1981. `http://www.springer.com/new+%26+forthcoming+titles+%28default%29/journal/493`.

[3] Larman and Rogers. The realization of distances within sets of Euclidean space. *Mathemaka*, 19:1–24, 1972.

[4] Raigorodskii. On the chromatic number of space. *Russian Math Surveys*, 55(2):351–352, 2000.

[5] A. Szlam. Monochromatic translates of configurations in the plane. *Journal of Combinatorial Theory-Series A*, 2001.