PROOF THREE of the Finite Canonical Ramsey Theorem: Mileti's SECOND Proof

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MODIFY PROOF TWO by GETTING RID of R_{a-1}.

Bill's Mantra

It is often easier to proof something harder.

Theorem: For all a, for all $\alpha \in \mathrm{N}$, for all k there exists n such that for all $COL : \binom{[n]}{a} \rightarrow \omega \times [\alpha]$ there exists a set H of size k such that

1. There exists $I \subseteq [a]$ such that H is I-homog with respect to $\Pi_1(COL)$.
2. H is homog with respect to $\Pi_2(COL)$.

Definition: $GER_a(k, \alpha)$ is the least n that works.

How to Use This

1. $GER_1(k, \alpha)$ EASY to bound (your HW)
2. Using modification of PROOF THREE can bound $GER_a(k, \alpha)$ using $GER_{a-1}(-,-)$ WITHOUT using R_{a-1} or any R at all!

Proof in the Style of Ramsey

Given $COL : \binom{[n]}{a} \rightarrow \omega \times [\alpha]$ define a sequence.
Stage $a-2$ $(\forall 1 \le i \le a-2)[x_i = i]$. $X = \{x_1, \ldots, x_{a-1}\}$.
$A_{a-1} = [n] - X$.
Stage s: Have $X = \{x_1, \ldots, x_{s-1}\}$,
$COL' : \binom{X}{a-1} \rightarrow \omega \times [\alpha] \times \{\mathrm{homog}, \mathrm{rain}\}$, and A_{s-1}.
KEY: Will use $GER_{a-1}(k', 2\alpha)$ for some k' later.
Let $A_s^0 = A_{s-1}$ and x_s be least element of A_{s-1}.

Construction

Form A_s^0, A_s^1, ..., $A_s^{\binom{s}{a-2}}$
Assume have A_s^{L-1} and $COL'(X_1, x_s)$, ..., $COL'(X_{L-1}, x_s)$ defined.
Notation: We denote A_s^L by A_L throughout.

THE REAL KEY DIFFERENCE

This is the KEY diff from PROOF TWO.
Before doing ANYTHING else we do the following: Let i be the number that MAXIMIZES

$$\{y \in A_{L-1} \mid \Pi_2(COL(X_L, x_s, y)) = i\}.$$

We ONLY work with these, we KILL all of the others. Let

$$A_0^{L-1} = \{y \in A_{L-1} \mid \Pi_2(COL(X_L, x_s, y)) = i\}.$$

Note

$$|A_0^{L-1}| \geq |A_{L-1}|/\alpha.$$

Case 1: $(\exists c)[|\{x \in A_0^{L-1} : COL(X_L, x_s, x) = c\}| \geq \sqrt{|A_0^{L-1}|}$.

$$\begin{aligned} COL'(X_L, x_s) &= (c, (i, \mathrm{homog})) \\ A_L &= \{x \in A_0^{L-1} : COL(X_L, x_s, x) = (c, i)\} \end{aligned}$$

Note: $|A_L| \geq \sqrt{|A_0^{L-1}|} \geq \sqrt{|A_{L-1}|/\alpha}$.

Can Ramsey Proof

Case 2: $(\forall c)[|\{x \in A_{L-1} : COL(X_L, x_s, x) = (c, i)\}| < \sqrt{|A_{L-1}|}$.
Make all colors coming out of (X_L, x_s) to the right different:

Let A_L be the set of all $x \in A_{L-1}$ such that x is the LEAST number with the color $COL(X_L, x_s, x)$.
Formally $A_L = \{x \in A_{L-1}$:

$$COL(X_L, x_s, x) \notin \{COL(X_L, x_s, y) : x_s < y < x \wedge y \in A_{L-1}\}$$

$$\}$$

Now have

$$(\forall y, y' \in A_L)[COL(X_L, x_s, y) \neq COL(X_L, x_s, y')].$$

Note: $|A_L| \geq \sqrt{|A_{L-1}|/\alpha}$.

Want to make colors DIFF

Important Note and Convention: For the rest of Case 2 we only care about $Z \in \binom{X}{a-1}$ such that $COL'(Z) = (-, (i, \mathrm{rain}))$.
Want to make the following true

$$(\forall Z \in \binom{X}{a-1})(\forall y, y' \in A_s)[COL(Z, y') \neq COL(X^L, y)]$$

Its OKAY if $COL(Z, y) = COL(X^L, y)$.

For each $y \in A_L$ we thin out A_L so that:

- $(\forall Z \in \binom{X}{a-1})(\forall y' \in A_L - \{y\})[COL(Z, y') \neq COL(X^L, y)]$.
- $(\forall Z \in \binom{X}{a-1})(\forall y' \in A_L - \{y\})[COL(Z, y) \neq COL(X^L, y')]$.

More to do!

Use C for COL for space

$T = A_L$ (elements to process)

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while T ≠ ∅
   y = least element of T.
   T = T − {y} (but y stays in A_L)
   If (∃Z ∈ (X choose a−1), y' ∈ T)[C(X_L, x_s, y) = C(Z, y')] then
      T = T − {y'}  A_L = A_L − {y'}
   If (∃Z ∈ (X choose a−1)) y' ∈ T)[C(X_L, x_s, y') = C(Z, y)] then
      T = T − {y'}  A_L = A_L − {y'}
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Note: At end $|A_L| \ge \sqrt{|A_{L-1}|/\sqrt{\alpha} s^{a-1}}$.

Note: At end

$(\forall Z \in \binom{X}{a-1}, y' \in A_L))[COL(X_L, x_s, y) \neq COL(Z, y')]$.

OKAY- What is $COL'(X_L, x_s)$?

RECAP:

- $(\forall y, y' \in A_L)[COL(X_L, x_s, y) \neq COL(X_L, x_s, y')]$
- $(\forall y, y' \in A_L)(\forall Z \in \binom{X}{a-1})[COL(X_L, x_s, x) \neq COL(Z, y')]$

$f(s)$ TBD. Let $t = |A_L| \geq \frac{\sqrt{|A_{L-1}|}}{\sqrt{\alpha s^{a-1}}}$

Case 2.1:
$(\exists Z \in \binom{X}{a-1})[|\{y : COL(X_L, x_s, y) = COL(Z, y)\}| \geq \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(X_L, x_s) &= COL(X_L, x_s) \\ A_L &= \{y \in A_L : COL(X_L, x_s, y) = COL(Z, y)\} \end{aligned}$$

Note: This will be a color of the form $(-, (i, \mathrm{rain}))$.
Note: $|A_L| \geq \frac{t}{f(s)}$.

OKAY- What is $COL'(X_L, x_s)$

Case 2.2:
$(\forall Z \in \binom{X}{a-1})[|\{y \in A_L : COL(X_L, x_s, y) = COL(Z, y)\}| < \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(X_L, x_s) &= (\ell, (i, \mathrm{rain}))\ \ell \text{ is least not-used-for-}\mathrm{rain}\text{ color.} \\ A_L &= A_L - \{y : (\exists Z \in \binom{X}{a-1})[COL(X_L, x_s, y) = COL(Z, y)]. \end{aligned}$$

Note: $|A_L| \geq t - \binom{s-1}{a-1}\frac{t}{f(s)} \geq t(1 - \binom{s-1}{a-1}\frac{1}{f(s)})$

Picking $f(s)$

Case 1 yields $|A_L| \geq \frac{t}{f(s)}$.

Case 2 yields $|A_L| \geq t(1 - \binom{s-1}{a-1}\frac{1}{f(s)})$

Take $f(s) = 1 + \binom{s-1}{a-1} \leq s^a/a!$. Both cases yield:

$$|A_L| \geq \frac{t}{f(s)} \geq \frac{\sqrt{|A_{L-1}|}}{\sqrt{\alpha}s^{a-1}}\frac{a!}{s^a} \geq c\sqrt{|A_{L-1}|}$$

where $c = \frac{a!}{\sqrt{\alpha}s^{2a}}$ (could have used s^{2a-1} but that would not gain us much).
We later see how far we need to go.

Whats Really Going on?

$c = \frac{a!}{\sqrt{\alpha}s^{2a}}$. We assume $c < 1$. (If $c \ge 1$ then we ignore it.)

$$\begin{aligned} b_0 &= b = a_{s-1} \\ b_L &\ge c\sqrt{b_{L-1}} \end{aligned}$$

By Rec Lemma

$$b_L \ge c^2 b^{1/2^L}.$$

In stage s do this for $\le s^{a-1}$ times. Hence

$$a_s \ge b_{s^{a-1}} \ge c^2 b^{1/2^{s^{a-1}}} \ge \frac{(a!)^2}{\alpha s^{4a}} b^{1/2^{s^{a-1}}} \ge dm^{1/2^{s^{a-1}}}$$

Where $d = \frac{(a!)^2}{s^{4a}\alpha}$.

Bound on A_s

Let $a_s = |A_s|$.

$$
\begin{aligned}
a_0 &= n \\
a_s &\ge d a_{s-1}^{1/2^{s^{a-1}}}.
\end{aligned}
$$

By Rec Lemma

$$a_s \ge d^2 n^{1/2^{s^a}}$$

We later see how far we need to go.

AVOID RAMSEY

We will run the construction until X has r elements— we determine r later
Have $X = \{x_1, x_2, \ldots, x_r\}$,
$COL' : \binom{X}{a-1} \rightarrow \omega \times ([\alpha] \times \{\mathrm{homog}, \mathrm{rain}\})$. We can apply $GER_{a-1}(k, 2\alpha)$
KEY: In PROOF TWO we applied Ramsey at this step to get either all homog or all rain. Here we don't need to since GER will take care of that.
Get I-homog set wrt to $\Pi_1 \circ COL'$ that is also homog wrt $\Pi_2 \circ COL'$.

$$H = \{z_1, z_2, \ldots, z_k\}.$$

Cases depend on if $\Pi_2 \circ COL'$ homog color is $(-, \mathrm{homog})$ or $(-, \mathrm{rain})$.

Case 1: Π_2 Color is $(-, \mathrm{homog})$ (real colors).
Have

$$(\forall Y \in \binom{H}{a-1}, z_1, z_2 \in H)[COL(Y, z_1) = COL(Y, z_2)].$$

H is I-homog set where $I \subseteq [a-1]$ wrt $\Pi_1 \circ COL'$.

$$COL(y_1, \ldots, y_a) = COL(z_1, \ldots, z_a) \text{ iff}$$

$$COL'(y_1, \ldots, y_{a-1}) = COL'(z_1, \ldots, z_{a-1})(\text{def of } COL' \text{ iff}$$

$$(\forall i \in I)[y_i = z_i](\text{def of } I\text{-homog})$$

So H is I-homog set.

Case 2: Π_2 Color is $(-, \mathrm{rain})$.
Have

$$(\forall Y \in \binom{H}{a-1}, z_1, z_2)[COL(Y, z_1) \ne COL(Y, z_2)].$$

H is I-homog set where $I \subseteq [a-1]$ wrt $\Pi_1 \circ COL'$.

$$COL(y_1, \ldots, y_a) = COL(z_1, \ldots, z_a) \text{ iff}$$

$$y_a = z_a \wedge COL'(y_1, \ldots, y_{a-1}) = COL'(z_1, \ldots, z_{a-1}) \text{ (from const.}$$

$$\text{iff } y_a = z_a \wedge (\forall i \in I)[y_i = z_i] \text{ (def of } I\text{-homog)).}$$

So H is $I \cup \{a\}$-homog set.
Need $r = GER_{a-1}(k, 2\alpha)$.

Estimate n

LET $r = GER_{a-1}(k, 2\alpha)$.
NEED

$$a_r \ge d^2 \frac{n^{1/2^{r^a}}}{r^{8a}} \ge 1$$

$$a_r \ge (a!/\alpha r^{4a}\alpha)^2 n^{1/2^{r^a}} \ge 1$$

$$n^{1/2^{r^a}} \ge \frac{e}{r}^{4a} \text{ where } e = \frac{1}{d^2} = \frac{\alpha^2}{(a!)^4}$$

$$n \ge er^{4a2^{r^a}}$$

If suffices to take
$n = \Gamma_2(ear^a)$

$GER_1(k,\alpha) = \alpha k^2$.
$GER_a(k) \le \Gamma_2(\frac{\alpha^2}{(a!)^4} \times a \times GER_{a-1}(k, 2\alpha))$
For simplicity lets not use the a. (The reader is challenged to get a better bound using it.
$GER_a(k) \le \Gamma_2(\alpha^2 GER_{a-1}(k, 2\alpha))$

$GER_1(k,\alpha) = \alpha k^2$.
$GER_1(k,2\alpha) = 2\alpha k^2$.

$GER_2(k,\alpha) \le \Gamma_2(\alpha^2 GER_1(k,2\alpha)) \le \Gamma_2(\alpha^2 2\alpha^2 k^2) = \Gamma_2(2\alpha^4 k^2)$
$GER_2(k,2\alpha) \le \Gamma_2(2(2\alpha)^4 k^2) = \Gamma_2(2^5\alpha^4 k^2)$

$GER_3(k,\alpha)) \le \Gamma_2(\alpha^2\Gamma_2(2^5\alpha^4 k^2) \le \Gamma_4(2^5\alpha^6 k^2)$
$GER_3(k,2\alpha)) \le \Gamma_4(2^{11}\alpha^6 k^2)$

$GER_4(k,\alpha)) \le \Gamma_2(\alpha^2\Gamma_4(2^{11}\alpha^6 k^2) \le \Gamma_6(2^{11}\alpha^8 k^2)$
$GER_4(k,2\alpha)) \le \Gamma_6(2^{19}\alpha^8 k^2)$

$GER_5(k,\alpha)) \le \Gamma_2(\alpha^2\Gamma_6(2^{19}\alpha^8 k^2)) \le \Gamma_8(2^{19}\alpha^{10} k^2)$
$GER_5(k,2\alpha)) \le \Gamma_8(2^{29}\alpha^{10} k^2)$

Can show

$$GER_a(k,\alpha) \le \Gamma_{2a-2}(2^{a+(a+1)^2}\alpha^{2a} k^2))$$

In particular:

$$ER_a(k) = GER_a(k,1) \le \Gamma_{2a-2}(2^{a+(a+1)^2} k^2)$$

PROS and CONS

1. GOOD-Proof reminsicent of Ramsey Proof.
2. GOOD-Seemed to be able to avoid alot of cases.
3. BAD-Proof complicated(?).
4. GOOD: $ER_a(k) \le \Gamma_{2a-2}(2^{a+(a+1)^2}k^2)$ BIG improvement!