The Complexity of Grid Coloring

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Grid Coloring

Notation: If $n \in \mathbb{N}$ then $[n]$ is the set $\{1, \ldots, n\}$.

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is c-colorable if there is a c-coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Examples

A FAILED 2-Coloring of $G_{4,4}$

R	B	B	R
B	R	R	B
B	B	R	R
R	R	R	B

A 2-Coloring of $G_{4,4}$

R	B	B	R
B	R	R	B
B	B	R	R
R	B	R	B

EXAMPLE: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that CANNOT 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

4-Colorability

1. Fenner, Gasarch, Glover, Purewall [FGGP] had reasons to think $G_{17,17}$ is 4-colorable but they did not have a 4-coloring.
2. In 2009 Gasarch offered a prize of \$289.00 for the first person to email him a 4-coloring of $G_{17,17}$.
3. Brian Hayes, Scientific American Math Editor, popularized the challenge.

Challenge Was Hard

1. Lots of people worked on it.
2. No progress.
3. Finally solved in 2012 by Bernd Steinbach and Christian Posthoff [SP]. Clever, and SAT-solver, but did not generalize.

Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere? YES!
2. Is there a Prop Statement about Grid Coloring whose resolution proof requires exp size? YES!

THERE IS AN NP-COMPLETE PROBLEM LURKING!

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \ldots, c\}$ is a *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $N \times M$.
2. $$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

   GCE is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
$\overline{x}_4$		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
$\overline{x}_3$		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
$\overline{x}_3$		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
$\overline{x}_2$		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
$\overline{x}_1$		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

Does this Explain why the Challenge was Hard?

1. MAYBE NOT: *GCE* is Fixed Parameter Tractable: For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
2. MAYBE NOT: Our result says nothing about the case where the grid is originally all blank.

Part II of Talk—Lower Bounds on Tree Resolution

YOU SAY YOU WANT A RESOLUTION!

Resolution

Definition

Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula. A Resolution Proof of $\varphi \notin SAT$ is a sequence of clauses such that on each line you have either

1. One of the C's in φ (called an AXIOM).
2. $A \vee B$ if $A \vee x$ and $B \vee \neg x$ were on prior lines. Variable that is resolved on is x.
3. The last line has the empty clause.

A Relevent Formula

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

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$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

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3. For $i, j, i', j' \in \{1, \ldots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

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The AND of the following:

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$$x_{ij1} \vee x_{ij2}.$$

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$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

   Interpretation: There is no mono 1-rectangle.

3. For $i, j, i', j' \in \{1, \ldots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

   Interpretation: There is no mono 2-rectangle.

A Relevent Formula

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$
$$x_{ij1} \vee x_{ij2}.$$
Interpretation: (i, j) is colored either 1 or 2.
2. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$
Interpretation: There is no mono 1-rectangle.
3. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$
Interpretation: There is no mono 2-rectangle.

We interpret this statement as saying

There is a 2-coloring of $G_{5,5}$.

This statement is known to be false.

GRID(n,m,c)

Definition

Let $n, m, c \in \mathbb{N}$. $GRID(n, m, c)$ is the AND of the following:

1. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$,

$$x_{ij1} \vee x_{ij2} \vee \cdots \vee x_{ijc}$$

   Interpretation: (i, j) is colored either 1 or $\cdots$ or c.

2. For $i, i' \in \{1, \ldots, n\}$, $j, j' \in \{1, \ldots, m\}$, $k \in \{1, \ldots, c\}$,

$$\neg x_{ijk} \vee \neg x_{i'jk} \vee \neg x_{ij'k} \vee \neg x_{i'j'k}$$

   Interpretation: There is no mono rectangle.

We interpret this statement as saying

There is a c-coloring of $G_{n,m}$.

NOTE: $GRID(n, m, c)$ has nmc VARS and $O(cn^2m^2)$ CLAUSES.

Our Goal

Assume that there is no c-coloring of $G_{n,m}$.

1. $GRID(n, m, c)$ has a size $2^{O(cnm)}$ Tree Res Proof.
2. We show $2^{\Omega(c)}$ size is REQUIRED. THIS IS OUR POINT!
3. The lower bound is IND of n, m.

Interesting Examples

1. Fenner et al [FGGP] showed that $G_{2c^2-c,,2c}$ is not c-colorable. Hence

$$GRID(2c^2 - c, 2c)$$

has $O(c^3)$ vars, $O(c^6)$ clauses but $2^{\Omega(c)}$ Tree Res proof.

2. Easy to show G_{c^3,c^3} is not c-colorable.

$$GRID(c^3, c^3, c)$$

has $O(c^7)$ vars, $O(c^{13})$ clauses and $2^{\Omega(c)}$ Tree Res proof.

These are poly-in-c formulas that require $2^{\Omega(c)}$ Tree Res proofs.

GRID(n,m,c) Requires Exp Tree Res Proofs

Theorem

Let n, m, c be such that $G_{n,m}$ is not c-colorable. Let $c \geq 2$.

1. *If $c \geq 2$ then any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.5c}$.*
2. *If $c \geq 9288$ then any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.836c}$.*

Technique: Use Prover-Delayer Games.

Open Questions

1. Want matching upper bounds for Tree Res Proofs of $GRID(n, m, c) \notin SAT$.
2. Want lower bounds on Gen Res Proofs of $GRID(n, m, c) \notin SAT$.
3. Want lower bounds on in other proof systems $GRID(n, m, c) \notin SAT$. (Have them for Tree cutting-Plane proofs.)

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