Two Proofs of the 3-Hypergraph Ramsey Theorem: An Exposition

William Gasarch-U of MD

3-Hypergraph Ramsey

Theorem

For all $COL : \binom{\mathsf{N}}{3} \rightarrow [2]$ *there exists an infinite homog set.*

Ramsey's Proof-Construction

$COL : \binom{\mathbb{N}}{3} \rightarrow [2]$.

CONSTRUCTION

$V_0 = \mathbb{N}$, $x_0 = 1$.

Assume V_{i-1} infinite, $x_1, x_2, \ldots, x_{i-1}$, $c_1, \ldots, c_{i-1}$ are all defined.

$$
\begin{aligned}
x_i &= \text{ the least number in } V_{i-1} \\
V_i &= V_{i-1} - \{x_i\} \text{ (We change this set w/o changing name.)} \\
COL^*(x,y) &= COL(x_i,x,y) \text{ for all } \{x,y\} \in \binom{V_i}{2} \\
V_i &= \text{ the largest 2-homog set for } COL^* \\
c_i &= \text{ the color of } V_i
\end{aligned}
$$

END OF CONSTRUCTION

KEY: for all $y, z \in V_i$, $COL(x_i, y, z) = c_i$.

Finish Ramsey Proof

We have vertices

$$x_1, x_2, \ldots,$$

and associated colors

$$c_1, c_2, \ldots,$$

There exists $i_1, i_2, \ldots$ such that

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

$$x_{i_1}, x_{i_2}, \ldots$$

is homog set.

PROS and CONS

1. PRO- proof was simple since it USED Ramsey's theorem.
2. CON- proof used Ramsey's theorem ω times and PHP once. Hence if want to finitize it get EEEEEEEENROMOUS bounds.

Erdos-Rado Proof

Same Theorem:

Theorem

For all $COL : \binom{N}{3} \rightarrow [2]$ *there exists an infinite homog set.*

CONSTRUCTION

$$\begin{aligned} x_1 &= 1 \\ V_1 &= \mathrm{N} - \{x_1\} \end{aligned}$$

Assume $x_1, \ldots, x_{i-1}$, V_{i-1}, and
$COL^{**} : \binom{\{x_1,\ldots,x_{i-1}\}}{2} \rightarrow \{\mathrm{RED}, \mathrm{BLUE}\}$ are defined.

$$\begin{aligned} x_i = &\quad \text{the least element of } V_{i-1} \\ V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this w/o changing its name).} \end{aligned}$$

We are NOT done yet. Next slide.

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, ..., $COL^{**}(x_{i-1}, x_i)$. We also define smaller sets V_i. We keep variable name V_i throughout.
For $j = 1$ to $i - 1$

1. $COL^* : V_i \rightarrow \{\text{RED}, \text{BLUE}\}$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.
2. Let V_i be redefined as an infinite homog set for COL^*.
3. $COL^{**}(x_j, x_i)$ is the color of V_i.

KEY: For all $1 \leq i_1 < i_2 \leq i$, for all $y \in V_i$,
$COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2})$.
END OF CONSTRUCTION

Finish it up

$$X = \{x_1, x_2, \ldots, \}$$

and $COL^{**} : \binom{X}{2} \rightarrow [2]$. Use RAMSEY's THEOREM ON GRAPHS!

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$

Easy to show H is homog.