1. (0 points) What is your name? When is the midterm? By what day
must you tell Dr. Gasarch you can’t make the midterm? (While this
problem is 0 points, if you miss the midterm and do not tell Dr.
Gasarch, you will get \(-100\) on every single homework problem 1).
When is the final?

2. (40 points) Recall the second proof of the infinite can Ramsey
theorem that used 3-ary, 4-color Ramsey and a maximal set argument. Finitize
it. Give a bound on \(\text{CR}_2(k)\), where you can have a Big-Oh in the
exponent.

(Note: You will learn how to do this in the Thurs Feb 27 lecture)

**SOLUTION TO PROBLEM TWO**

We first state the finite version of the 1-ary can Ramsey theorem, which
is from a previous homework: For every coloring of \((k - 1)^2 + 1\) points,
there exists a set of \(k\) points that are all the same color or a set of \(k\)
points that are all different colors. That is, \(\text{CR}_1(k) = (k - 1)^2 + 1\).

Let \(\text{COL}: \binom{[n]}{2} \to \omega\) be given. Define \(\text{deg}_c(v)\) for \(v \in [n]\) as we did in
class.

Let \(X\) be any set with a coloring \(\binom{X}{2} \to \omega\), and let \(R\) be a maximal
rainbow set of \(X\). Suppose \(\text{deg}_c(x) \leq 1\) for all \(x \in X\).

We define \(f\) as a function that maps \(y \in X \setminus R\) to the reason \(y \notin R\).
The reason \(y \notin R\) could be:

- Case 1: \(\exists u \in R, \exists \{a, b\} \in \binom{R}{2}\), s.t. \(\text{COL}(u, y) = \text{COL}(a, b)\).
- Case 2: \(\exists a \in R, \exists b \in R\), s.t. \(\text{COL}(u, a) = \text{COL}(u, b)\). This can’t
  happen because \(\text{deg}_c(u) \leq 1\) for all \(c\).

So only Case 1 can happen. Then, define \(f: X \setminus R \to R \times \binom{R}{2}\) where
\(f(y) = (u, \{a, b\})\) means that \(y \notin R\) because \(\text{COL}(u, y) = \text{COL}(a, b)\).

Note that \(f\) is injective, so we have \(|X| - |R| \leq |R| \times \frac{1}{2} |\binom{R}{2}|\), so we have
roughly \(|X| \leq |R|^3 + |R|\), or \(|R| \geq |X|^{1/3}\).

So our rainbow set has size at least \(|X|^{1/3}\).
Next we define \( \text{COL}' : \binom{[n]}{3} \to [4] \) from \( \text{COL} \) as we did in class:

\[
\text{COL}'(x_1, x_2, x_3) = \begin{cases} 
1 & \text{if } \text{COL}(x_1, x_2) = \text{COL}(x_1, x_3) \\
2 & \text{if } \text{COL}(x_1, x_3) = \text{COL}(x_2, x_3) \\
3 & \text{if } \text{COL}(x_1, x_2) = \text{COL}(x_2, x_3) \\
4 & \text{otherwise}
\end{cases}
\]

where each case assumes the negation of the previous case, and \( x_1 < x_2 < x_3 \).

We apply 3-ary Ramsey to get a homogeneous set \( H \) for \( \text{COL}' \). Using the bound \( R_3(k) \leq 2^{2^k} \), we get that \( |H| \geq \frac{1}{4} \log \log n \). There are four possible colors \( H \) can be:

- **Case 1:** If \( H \) has color 1, we have that for all \( x, y, z \in H \) with \( x < y < z \), \( \text{COL}(x, y) = \text{COL}(x, z) \). So, we define \( \text{COL}'' : H \to \omega \) so that \( \text{COL}''(x) \) is the color that is equal to \( \text{COL}(x, y) \) for all \( y \in H \) with \( x < y \). Now we can apply the finite 1-ary can Ramsey to get a set \( H' \) that is either rainbow or homogeneous. If \( H' \) is homog for \( \text{COL}'' \), then \( H' \) is homog for \( \text{COL} \). On the other hand, if \( H' \) is rainb for \( \text{COL}'' \), then it is min-homog for \( \text{COL} \). By the finite 1-ary can Ramsey theorem, our min-homog or homog set has size \( |H'| \geq \sqrt{|H|} - 1 \geq \sqrt{\frac{1}{4} \log \log n} - 1 \).

- **Case 2:** If \( H \) has color 2, it is similar to Case 1. We define \( \text{COL}''(y) \) to be the color that is \( \text{COL}(x, y) \) for all \( x < y \). Applying can Ramsey gives \( H' \) which is either rainb in \( \text{COL}'' \) and max-homog in \( \text{COL} \), or homogeneous in \( \text{COL}'' \) and also homog in \( \text{COL} \). The size of the homog or max-homog set is the same as the previous case, \( |H'| \geq \sqrt{\frac{1}{4} \log \log n} - 1 \).

- **Case 3:** If \( H \) has color 3, then a similar argument to the one from class shows that \( H \) is homog for \( \text{COL} \). Our homog set has size \( |H| \geq \frac{1}{4} \log \log n \).

- **Case 4:** If \( H \) has color 4, then none of the above cases hold. This is the case where for all \( x \) and for all \( c, \deg_c(x) \leq 1 \). By the reasoning given above, there is a maximal rainbow set \( R \subseteq H \) with size \( |R| \geq |H|^{1/3} \geq (\frac{1}{4} \log \log n)^{1/3} \).
Thus, we have a homog, min-homog, or max-homog set of size at least $k = (\frac{1}{4} \log \log n)^{1/3}$ (the smallest of the bounds above). Solving for $n$ gives:

$$n = 2^{2^{4k^3}}$$

Specifically, this means if we have $n = 2^{2^{4k^3}}$, we will have a homog, min-homog, max-homog, or rainb set of size $k$. Our final bound is then:

$$\text{CR}_2(k) \leq 2^{2^{O(k^3)}}$$

**END OF SOLUTION TO PROBLEM 3**

3. (40 points) The $n \times m$ grid is the set of points

$$\{(a, b) : 1 \leq a \leq n \text{ and } 1 \leq b \leq m\}.$$ 

In this problem we will be coloring these points.

A *monochromatic rectangle* is when there are FOUR points that are the corners of a rectangle that are all the same color. Example would be

$$\{(3, 4), (3, 8), (7, 4), (7, 8)\}.$$ 

For which values of $m$ can the $4 \times m$ grid be 3-colored without having a monochromatic rectangle? Prove your result.

**THERE IS ANOTHER PAGE TO THIS HW**

**SOLUTION TO PROBLEM FOUR**

Omitted, will do in class.

**END OF SOLUTION TO PROBLEM FOUR**
4. (20 points) Complete the following statement of a theorem so that it is correct and then prove it:

For all $\text{COL}$: $\left(\mathbb{N}^\mathbb{N}\right) \rightarrow \omega$, there exists an infinite set $H$ such that either: $\text{BLAH}$, or $\text{BLAH}$, or ..., or $\text{BLAH}$.

5. (0 points but you must do this so we can discuss) Here is a book review of a book on the Banach-Tarski Paradox:


Read the review. Be prepared to discuss if you think the BT paradox is TRUE or FALSE or SOMETHING ELSE. There is no right answer here but I want to know what you think.

6. (0 points) Compare and contrast the following parodies of Billy Joel’s *The Longest Time*:

- “The Longest Path” https://www.youtube.com/watch?v=a3ww0gwEszo
- “Entropic Time” https://www.youtube.com/watch?v=i6rVHr60wjI (does the singer look like anybody you know?)
- “Graduate on Time” https://www.youtube.com/watch?v=Vw6h6epNS5k
- “Polynomial Time” https://www.youtube.com/watch?v=o09nF0o8q-

For reference, here is the original: https://www.youtube.com/watch?v=a_XgQhMPeEQ