

1 Conjecture

In this project, we will consider the following conjecture:

For any \( c \in \mathbb{N} \), there exists a number \( E = E(c) \) such that for all \( c \)-colorings of \( \{1, 2, 3, \ldots, E\} \), there exists \( x, y, z \) such that:

- \( x, y, z \) are the same color (I bet you saw that coming!), and
- \( x^2 + y^2 = z^2 \)

The conjecture is known to be true for \( c = 1 \) (this is trivial) and for \( c = 2 \) (this is not so trivial).

We will gather evidence for how big \( E \) might be.

2 Greedy Algorithm

To find lower bounds on \( E(c) \), we find a number \( n \) and a \( c \)-coloring of \( [n] = \{1, 2, 3, \ldots, n\} \) such that there is no monochromatic triple \( x, y, z \) such that \( x^2 + y^2 = z^2 \). We will call such a \( c \)-coloring of \( \{1, 2, \ldots, n\} \) a valid coloring. If a valid \( c \)-coloring exists for \( \{1, 2, \ldots, n\} \) we will say that \( [n] \) can be \( c \)-colored.

We will consider the following greedy algorithm for finding valid colorings:

For each number \( k \) starting from 1, color \( k \) with the least color possible. That is, assign \( k \) the least color \( \chi \) from the set

\[
\{ \chi : \forall x, y < k \text{ s.t. } \text{COL}(x) = \text{COL}(y) = \chi \} \cup \{ x^2 + y^2 \neq k^2 \}\]

Keep coloring points as long as possible, until you reach a number \( y \) that can’t be colored without creating a monochromatic \( x, y, z \) with \( x^2 + y^2 = z^2 \).

For example, this approach would end up coloring \( \text{COL}(1) = \text{COL}(2) = \text{COL}(3) = \text{COL}(4) = 1; \) then coloring \( \text{COL}(5) = 2 \) to avoid \( 3, 4, 5 \) all being the same color.
3 Randomized (Greedy) Algorithm

Consider the following modification to the Greedy algorithm from the last section: When coloring a number \( k \), consider all the valid colors available for color \( k \), and pick one of these at random.

That is, randomly pick one color from the set

\[
\{ \chi : (\forall x, y < k \text{ s.t. } \text{COL}(x) = \text{COL}(y) = \chi)[x^2 + y^2 \neq k^2] \}
\]

and assign \( k \) this color.

As with the original greedy, we continue until some number can’t be colored (i.e. the set of valid colors above is empty).

4 The Project

1. (a) Write a program that implements the regular greedy algorithm and outputs a number \( n \), and a coloring of \( \{1, 2, \ldots, n\} \).

   (b) Write a program that implements the randomized greedy algorithm. It will also output a number \( n \) and a coloring of \( \{1, 2, \ldots, n\} \).

   Submit your code.

2. Run the greedy algorithm for \( c = 2 \) to find a number \( n \) and a 2-coloring of \( \{1, 2, \ldots, n\} \) with no \( x, y, z \) such that \( x^2 + y^2 = z^2 \) and \( x, y, z \) are all the same color.

   What number \( n \) does the greedy algorithm find and output?

   Run the randomized greedy algorithm for \( c = 2 \) to find a number \( n \) such that \( \lceil n \rceil \) can be 2-colored. Run it 50 times. What is the largest \( n \) that the randomized greedy algorithm output?

3. Run the greedy algorithm for \( c = 3 \). What number \( n \) does the greedy algorithm find and output?

   Run the randomized greedy algorithm 50 times, for \( c = 3 \). What is the largest \( n \) that the randomized greedy algorithm output?

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4. Run the greedy algorithm for $c = 4$. What number $n$ does the greedy algorithm find and output?

Run the randomized greedy algorithm 50 times, for $c = 4$. What is the largest $n$ that the randomized greedy algorithm output?

5. Run the greedy algorithm for $c = 5$. What number $n$ does the greedy algorithm find and output?

Run the randomized greedy algorithm 50 times, for $c = 5$. What is the largest $n$ that the randomized greedy algorithm output?

(Warning: Nathan’s code took 3 minutes to run 50 times, so yours might take a little while as well)