The Infinite Can Ramsey Theorem

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From the 1950 “Kürschák/Eötvös Math Competition”:

There are 1950 cans of paint. Find an \( x \) such that (1) there are either \( x \) cans of paint all the same color, or \( x \) cans of paint that are all different colors and (2) it is possible to have neither \( x + 1 \) cans that are all the same nor \( x + 1 \) cans that are all different.
There are 1950 cans of paint. Find an $x$ such that (1) there are either $x$ cans of paint all the same color, or $x$ cans of paint that are all different colors and (2) it is possible to have neither $x + 1$ cans that are all the same nor $x + 1$ cans that are all different.

From Homework, you know the answer is $\lfloor \sqrt{1949} \rfloor + 1 = 45$. 
Can Ramsey Theorem

The Can Ramsey Theorem deals with when we allow any number of colors.

It is named “Can Ramsey” in honor of the paint can problem on the 1950 Kürschák/Eötvös Math Competition
1-ary Ramsey’s Theorem

**Theorem:** For every $COL : \mathbb{N} \rightarrow [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necessarily get a homog set since could color EVERY vertex differently. But then get infinite *rainbow set.*
**Theorem:** Let $V$ be a countable set. Let $COL : V \to \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).
Theorem: Let $V$ be a countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

Prove with your neighbor.
Ramsey’s Theorem For Graphs

**Theorem:** For every $COL : \binom{\mathbb{N}}{2} \to [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite *rainbow set.*
Theorem: For every $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainb set.
VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.
Theorem: For every $COL: \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainb set.

VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.

FALSE:

- $COL(i, j) = \min\{i, j\}$.
- $COL(i, j) = \max\{i, j\}$.
Min-Homog, Max-Homog, Rainbow

**Definition:** Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- $V$ is homogenous if $COL(a, b) = COL(c, d)$ iff TRUE.
- $V$ is min-homogenous if $COL(a, b) = COL(c, d)$ iff $a = c$.
- $V$ is max-homogenous if $COL(a, b) = COL(c, d)$ iff $b = d$.
- $V$ is rainb if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

**Can Ramsey Theorem for $\binom{\mathbb{N}}{2}$:** For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set $V$ such that either $V$ is homog, min-homog, max-homog, or rainb.
Proof of Can Ramsey Theorem for $\binom{\mathbb{N}}{2}$

We are given $COL : \binom{\mathbb{N}}{2} \to \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use $COL$ to define $COL' : \binom{\mathbb{N}}{4} \to [16]$
We then apply 4-ary Ramsey theorem. (an “Application!”)

In the slides below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases.

Homog always means infinite Homog.
Pairs that begin the same way are same color

1. $COL(x_1, x_2) = COL(x_1, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 1$.
2. $COL(x_1, x_2) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 2$.
3. $COL(x_1, x_3) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 3$.
4. $COL(x_2, x_3) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 4$.

$H$ is homog set, color 1 (rest similar)

$COL'' : H \rightarrow \omega$ is $COL''(x) =$ color of all $(x, y)$ with $x < y \in H$.

Use 1-dim Can Ramsey!:

Case 1: $COL''$ has homog set $H'$ then $H'$ homog for COL.
Case 2: $COL''$ has rainb set $H'$ then $H'$ min-homog for COL.
Pairs that End the same way are same color

1. $COL(x_1, x_3) = COL(x_2, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 5.$

2. $COL(x_1, x_4) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 6.$

3. $COL(x_1, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 7.$

4. $COL(x_2, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 8.$

$H$ is homog set, color 5 (rest similar)

$COL'' : H \rightarrow \omega$ is $COL''(y) = \text{color of all } (x, y) \text{ with } x < y \in H.$

Use 1-dim Can Ramsey!:

Case 1: $COL''$ has homog set $H'$ then $H'$ homog for COL.

Case 2: $COL''$ has rainb set $H'$ then $H'$ max-homog for COL.
Easy Homog Cases

1. $COL(x_1, x_2) = COL(x_2, x_3) \Rightarrow COL(x_1, x_2, x_3, x_4) = 9.$
2. $COL(x_1, x_2) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 10.$
3. $COL(x_1, x_2) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 11.$
4. $COL(x_1, x_3) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 12.$
5. $COL(x_1, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 13.$
6. $COL(x_2, x_3) = COL(x_1, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 14.$
7. $COL(x_2, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 15.$

$H$ is homog set, color 9 (rest similar)

For all $w < x < y < z \in H$.

$$COL(w, x) = COL(x, y) = COL(y, z).$$

Other cases, like $COL(w, y) = COL(x, z)$, are similar
Rainbow Case

If **NONE** of the above cases hold then $COL(x_1, x_2, x_3, x_4) = 16$.

Let $H$ be homog set.

All edges from $H$ diff colors, so Rainbow Set.
PROS and CONS of Proof

**PRO:** Each Case easy. Note that Rainbow case was easy.

**CON:** Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.

Let $\text{CR}_2(k) = \text{least } n \text{ s.t. } \forall \text{COL: } ([n]) \to \omega, \exists H \text{ of size } k \text{ such that either } H \text{ is homog, min-homog, max-homog, or rainb. If finitized, this proof obtains}$

$$\text{CR}_2(k) \leq R_4(k, 16) \leq 16^{16^{O(k)}}$$
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\[
CR_2(k) \leq R_4(k, 16) \leq 16^{16^{16^{O(k)}}}
\]

We will give another proof which only uses 3-ary hypergraph Ramsey.
Definition Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. If $c$ is a color and $v \in \mathbb{N}$ then $\deg_c(v)$ is the number of $c$-colored edges with $v$ as an endpoint.

Note: $\deg_c(v)$ could be infinite.
**Lemma** Let $X$ be infinite. Let $COL : \binom{X}{2} \to \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \leq 1$ then there is an infinite rainb set. TRY TO PROVE WITH YOUR NEIGHBOR. I WILL THEN GIVE PROOF.
Proof

Let $R$ be a MAXIMAL rainb set of $X$.

$$(\forall y \in X - R)[R \cup \{y\} \text{ is not a rainb set}].$$

We prove $R$ is infinite.
Proof that $R$ is infinite

Let $y \in X - R$. Why is $y \notin R$?

1. $(\exists u \in R, \exists \{a, b\} \in \binom{R}{2})[\text{COL}(y, u) = \text{COL}(a, b)]$.

2. $(\exists \{a, b\} \in \binom{R}{2})[\text{COL}(y, a) = \text{COL}(y, b)]$.

   If $c = \text{COL}(y, a)$ then $\deg_c(y) \geq 2$, so Can’t Happen!

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a, b\})$ (item 1).

Map is injective: if $y_1$ and $y_2$ both map to $(u, \{a, b\})$ then $\text{COL}(y_1, u) = \text{COL}(y_2, u)$ but $\deg_c(u) \leq 1$.

Injection from $X - R$ to $R \times \binom{R}{2}$. If $R$ finite then injection from an infinite set to a finite set Impossible! Hence $R$ is infinite.
Can Ramsey Theorem for $\mathbb{N}$

**Theorem:** For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is either
- an infinite homogenous set,
- an infinite min-homog set,
- an infinite max-homog set, or
- an infinite rainb set.
Proof of Can Ramsey Theorem for Graphs

Given $COL : \binom{\mathbb{N}}{2} \to \omega$. We use $COL$ to obtain $COL' : \binom{\mathbb{N}}{3} \to [4]$. We will use the 3-ary Ramsey theorem. In all of the below $x_1 < x_2 < x_3$.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1 < x_2 < x_3) = 1$.
2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 2$.
3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 3$.
4. If none of the above occur then $COL'(x_1 < x_2 < x_3) = 4$.

Cases 1,2,3 are just like in the prior proof.

Case 4: For all $x$, for all $c$, $\deg_c(x) \leq 1$ so have Rainbow by Lemma.
Better Bounds on Can Ramsey

Using 4-ary proof, 16 colors, bound was:

\[ CR_2(k) \leq 16^O(k) \]

Using new proof, 3-ary with 4 colors, bound is:

Not obvious! Cases 1, 2, and 3 easy, but case 4 uses maximal sets.

Good news: Will be on homework.
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Better Bounds on Can Ramsey

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