The Infinite Can Ramsey Theorem (An Exposition)

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1-ary Ramsey’s Theorem

**Theorem:** For every $COL : N \rightarrow [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necessarily get a homog set since could color EVERY vertex differently. But then get infinite *rainbow set*. 
Theorem: Let $V$ be a countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).
One-Dim Can Ramsey Theorem

**Theorem:** Let $V$ be a countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

Prove with your neighbor.
Ramsey’s Theorem For Graphs

**Theorem:** For every $COL : \binom{\mathbb{N}}{2} \to [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*. 
Attempt

**Theorem:** For every $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainb set.

**VOTE:** TRUE, FALSE, or UNKNOWN TO SCIENCE.
**Theorem:** For every $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainb set.

**VOTE:** TRUE, FALSE, or UNKNOWN TO SCIENCE.

**FALSE:**

- $COL(i, j) = \min\{i, j\}$.
- $COL(i, j) = \max\{i, j\}$. 
Definition: Let $COL : \binom{N}{2} \rightarrow \omega$. Let $V \subseteq N$.

- $V$ is homogenous if $COL(a, b) = COL(c, d)$ iff TRUE.
- $V$ is min-homogenous if $COL(a, b) = COL(c, d)$ iff $a = c$.
- $V$ is max-homogenous if $COL(a, b) = COL(c, d)$ iff $b = d$.
- $V$ is rainb if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

Can Ramsey Theorem for $\binom{N}{2}$: For all $COL : \binom{N}{2} \rightarrow \omega$, there exists an infinite set $V$ such that either $V$ is homog, min-homog, max-homog, or rainb.
Proof of Can Ramsey Theorem for $\binom{\mathbb{N}}{2}$

We are given $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use $COL$ to define $COL' : \binom{\mathbb{N}}{4} \rightarrow [16]$
We then apply 4-ary Ramsey theorem. (an “Application!”)

In the slides below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases.

Homog always means infinite Homog.
Pairs that begin the same way are same color

1. $COL(x_1, x_2) = COL(x_1, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 1$.
2. $COL(x_1, x_2) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 2$.
3. $COL(x_1, x_3) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 3$.
4. $COL(x_2, x_3) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 4$.

$H$ is homog set, color 1 (rest similar)

$COL'' : H \rightarrow \mathbb{N}$ is $COL''(x) =$ color of all $(x, y)$ with $x < y \in H$.

Use 1-dim Can Ramsey!:

Case 1: $COL''$ has homog set $H'$ then $H'$ homog for $COL$.
Case 2: $COL''$ has rainb set $H'$ then $H'$ min-homog for $COL$. 
Pairs that End the same way are same color

1. \( \text{COL}(x_1, x_3) = \text{COL}(x_2, x_3) \rightarrow \text{COL}'(x_1 < x_2 < x_3 < x_4) = 5. \)
2. \( \text{COL}(x_1, x_4) = \text{COL}(x_2, x_4) \rightarrow \text{COL}'(x_1 < x_2 < x_3 < x_4) = 6. \)
3. \( \text{COL}(x_1, x_4) = \text{COL}(x_3, x_4) \rightarrow \text{COL}'(x_1 < x_2 < x_3 < x_4) = 7. \)
4. \( \text{COL}(x_2, x_4) = \text{COL}(x_3, x_4) \rightarrow \text{COL}'(x_1 < x_2 < x_3 < x_4) = 8. \)

\( H \) is homog set, color 5 (rest similar)

\( \text{COL}'' : H \rightarrow \mathbb{N} \) is \( \text{COL}''(y) = \text{color of all } (x, y) \text{ with } x < y \in H. \)

Use 1-dim Can Ramsey!:

Case 1: \( \text{COL}'' \) has homog set \( H' \) then \( H' \) homog for \( \text{COL} \).

Case 2: \( \text{COL}'' \) has rainb set \( H' \) then \( H' \) max-homog for \( \text{COL} \).
Easy Homog Cases

1. \( \text{COL}(x_1, x_2) = \text{COL}(x_2, x_3) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 9. \)
2. \( \text{COL}(x_1, x_2) = \text{COL}(x_2, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 10. \)
3. \( \text{COL}(x_1, x_2) = \text{COL}(x_3, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 11. \)
4. \( \text{COL}(x_1, x_3) = \text{COL}(x_2, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 12. \)
5. \( \text{COL}(x_1, x_3) = \text{COL}(x_3, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 13. \)
6. \( \text{COL}(x_2, x_3) = \text{COL}(x_1, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 14. \)
7. \( \text{COL}(x_2, x_3) = \text{COL}(x_3, x_4) \Rightarrow \text{COL}(x_1, x_2, x_3, x_4) = 15. \)

\( \text{H is homog set,color 9 (rest similar)} \)

For all \( w < x < y < z \in H. \)

\[ \text{COL}(w, x) = \text{COL}(x, y) = \text{COL}(y, z). \]

Other cases, like \( \text{COL}(w, y) = \text{COL}(x, z) \), are similar
If **NONE** of the above cases hold then $COL(x_1, x_2, x_3, x_4) = 16$.

Let $H$ be homog set.

All edges from $H$ diff colors, so Rainbow Set.
PROS and CONS of Proof

**PRO:** Each Case easy. Note that Rainbow case was easy.

**CON:** Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.

Let $\text{CR}_2(k) =$ least $n$ s.t. $\forall \text{COL}: \binom{N}{2} \rightarrow \omega$, $\exists H$ of size $k$ such that either $H$ is homog, min-homog, max-homog, or rainb. If finitized, this proof obtains

$$\text{CR}_2(k) \leq R_4(k, 16) \leq 16^{16^{16^{O(k)}}}$$
PROS and CONS of Proof

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$$CR_2(k) \leq R_4(k, 16) \leq 16^{16^{16^{O(k)}}}$$

We will give another proof which only uses 3-ary hypergraph Ramsey.
**Definition** Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. If $c$ is a color and $v \in \mathbb{N}$ then $\deg_c(v)$ is the number of $c$-colored edges with $v$ as an endpoint.

**Note:** $\deg_c(v)$ could be infinite.
Lemma Let $X$ be infinite. Let $COL : \binom{X}{2} \to \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \leq 1$ then there is an infinite rainb set. TRY TO PROVE WITH YOUR NEIGHBOR. I WILL THEN GIVE PROOF.
Proof

Let $R$ be a MAXIMAL rainb set of $X$.

$$(\forall y \in X - R)[X \cup \{y\} \text{ is not a rainb set}].$$

Let $y \in X - R$. Why is $y \notin R$?

1. $(\exists u \in R, \exists \{a, b\} \in \binom{R}{2})[\text{COL}(y, u) = \text{COL}(a, b)]$.

2. $(\exists \{a, b\} \in \binom{R}{2})[\text{COL}(y, a) = \text{COL}(y, b)]$.

If $c = \text{COL}(y, a)$ then $\deg_c(y) \geq 2$, so Can’t Happen!

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a, b\})$ (item 1).

Map is injective: if $y_1$ and $y_2$ both map to $(u, \{a, b\})$ then $\text{COL}(y_1, u) = \text{COL}(y_2, u)$ but $\deg_c(u) \leq 1$.

Injection from $X - R$ to $R \times \binom{R}{2}$. If $R$ finite then injection from an infinite set to a finite set Impossible! Hence $R$ is infinite.
Theorem: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is either

▶ an infinite homogenous set,
▶ an infinite min-homog set,
▶ an infinite max-homog set, or
▶ an infinite rainb set.
Proof of Can Ramsey Theorem for Graphs

Given \( COL : \binom{\mathbb{N}}{2} \to \omega \). We use \( COL \) to obtain \( COL' : \binom{\mathbb{N}}{3} \to [4] \). We will use the 3-ary Ramsey theorem. In all of the below \( x_1 < x_2 < x_3 \).

1. If \( COL(x_1, x_2) = COL(x_1, x_3) \) then \( COL'(x_1 < x_2 < x_3) = 1 \).
2. If \( COL(x_1, x_3) = COL(x_2, x_3) \) then \( COL'(x_1 < x_2 < x_3) = 2 \).
3. If \( COL(x_1, x_2) = COL(x_2, x_3) \) then \( COL'(x_1 < x_2 < x_3) = 3 \).
4. If none of the above occur then \( COL'(x_1 < x_2 < x_3) = 4 \).

Cases 1, 2, 3 are just like in the prior proof. Case 4: For all \( x \), for all \( c \), \( \deg_c(x) \leq 1 \) so have Rainbow by Lemma.
There is an infinite homog set of color 4: Recall: all pairs of \( x_1, x_2, x_3 \) have diff colors. Let \( H \) be the infinite homog set.

Rename so

\[
H = \{1, 2, 3, \ldots\}
\]

**GOOD NEWS:** (1, 2) and (2, 3) diff colors.

**BAD NEWS:** (1, 2) and (3, 4) could be same color.

**USEFUL NEWS:** Let \( RE \) be the set of all RED edges. The set \( RE \) is a set of disjoint edges.

CANNOT have, say (4,100) and (100,200) in \( RE \).

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Need to do some more killing!
Case 4 cont:

Lets out all edges in order of max number:

(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (4, 5)...

We process each edge. 
(1,2): Say its RED. We want to KILL all RED edges but still have an infinite number of vertices. Let 
(a₁, b₁), (a₂, b₂), ... be all the RED edges. KEY: all disjoint and none have 1 or 2 in them. Assume aᵢ < bᵢ. 
KILL ALL THE bᵢ’s!
Look at the next edge on the list thats left. Do the same. 
When done have bloody rainbow set!