CMSC 858M: Algorithmic Lower Bounds: Fun With Hardness Proofs
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Instructor: Mohammad T. Hajiaghayi
Scribe: Alexander Mendelsohn

1 Overview

This chapter discusses multiple variations of SAT. It talks about which ones are in P and which in NP, and uses them to prove other problems are NP-complete.

2 Variants of SAT

Deciding whether a boolean formula in conjunctive normal form (CNFSAT, or just SAT for short) is an NP-hard problem. Circuit satisfiability (deciding whether a circuit can have inputs that produce the output 1) is also NP-hard. DNFSAT (like CNFSAT but in disjunctive normal form) is in P. Converting a CNFSAT problem to a DNFSAT problem to solve it is not practical in some cases because it can take an exponential number of clauses.

The most common SAT problem is 3SAT, in which each clause has ≤ 3 variables. 3SAT is NP-complete, though 2SAT is in P.

Definition 1 *Here are some variants on SAT problems.* $\mathtt{a}, \mathtt{b}$ *are assumed to be natural numbers.*

1. *aSAT: each clause has* $\leq \mathtt{a}$ *vaiables.*
2. *EaSAT: each clause has exactly a variables.*
3. *EUaSAT: each clause has exactly a variables and each variable occurs at most once per clause.*
4. *SAT-b: each variable occurs at most b times.*
5. *NAE-SAT: within each clause the variables can neither be all true nor all false. In this problem, true and false are symmetric.*

6. *1-IN-aSAT: exactly 1 variable in each clause is true (can replace 1 with any number).*
7. *MONO-SAT: a formula is monotone if either all variables in all clauses are positive or all are negative.*
8. *MAX-SAT: the problem is not to satisfy all of the clauses, just to satisfy as many as possible.*
9. *HORNSAT: each clause has ≤ 1 positive variable.*
10. *DUAL HORNSAT: each clause has ≤ 1 negative variable.*
11. *RENAMABLE HORNSAT: if you replace every variable in a subset of clauses with a negation then the result is HORNSAT.*
12. *Many of these prefixes and suffixes can be combined at will.*

3 Some Results

3SAT-3 is NP-complte.
EU3SAT-3 is always satisfiable so it is trivialy in P.
MONO-3SAT is NP-Complete.
MONO-3SAT-4 is NP-Complete.
MAX-2SAT is NP-hard.
HORNSAT and DUAL HORNSAT are both in P.
1-in-3SAT is NP-complete.
MONO 1-in-3SAT is NP-complete.
MONO-NAE-3SAT is NP-complete.

Schaefer's Dichotomy Theorem says every version of SAT is either in P, or NP-Complete. It also give a set of six conditions that make a SAT-type problem in P. It it satisfies none of those conditions than it is NP-hard.

Definition 2 *A Relation on m variables is a truth table or formula on those literals. A SAT-type problem is a set of relations.*

Theorem 1 *Schaefer's Dichotomy Theorem: Let $R_1, ..., R_k$ be a set of relations. The problem is in P if and only if one of the following conditions holds. Otherwise it is NP-complete.*

1. *For all* $1 \leq i \leq k, R_i(T, ..., T) = T$
2. *For all* $1 \leq i \leq k, R_i(T, ..., T) = F$
3. *For all* $1 \leq i \leq k, R_i(x_1, ..., x_m)$ *is equivalent to a conjunction of clauses with 1 or two variables.*

4. *For all* $1 \leq i \leq k, R_i(x_1, ..., x_m)$ *is equivalent to a Horn clause.*
5. *For all* $1 \leq i \leq k, R_i(x_1, ..., x_m)$ *is equivalent to a dual-Horn Clause.*
6. *For all* $1 \leq i \leq k, R_i(x_1, ..., x_m)$ *is equivalent to a system of linear equations over mod 2.*

2-Colorable Perfect Matching is the question of whether you can 2-color a graph such that every vertex has exactly one neighbor of the same color. This is shown to be NP-complete, even when restricted to planar 3-regurlar graphs, by a reduction from MONO-NAE-3SAT.

4 Some Games

Cryptarithms can be proven NP-complete by a reduction from MONO-1-in-3SAT.

Various versions of pushblock games are either NP-complete or PSPACE-complete.

Mario Brothers, Donkey Kong, Legends of Zelda, Metroid, and Pokemon are NP-hard.
Conway's Phutball is PSLACE-hard, and even just checking whether the position is a mate-in-1 is NP-hard.
Deciding whether a checkers position (generalized to an arbitrarily large board) is mate-in-1 is in P. Deciding whether a player can force the other playet to win in one move is NP-complete. Checkers where jumping is mandatory is PSPACE-complete.
Deciding whether a crease-pattern with orientation has a flat fold origami is NP-hard.
Vertex Disjoing Paths is the question of, given a graph and pairs of start and end points, are there paths between each pair that don't share any vertices. This problem is NP-complete. So are the planar and rectangle versions.

5 Other NP-Complete Problems

The problem Spiral Galaxies, also known as Tentai Show, can be shown to be NP-complte via a reduction for Circuit SAT. [**?**].

The problem of 0-1 Integer Programming is: Given a matrix $M \in \mathbb{Z}^{m \times n}$ and a vector $V \in \mathbb{Z}^m$, is there a vector $x \in \{0, 1\}^n$ such that $Mx \leq V$. This problem can be shown to be NP-complete by reducing from 3SAT.

Given a rectangular board, with each square containing a red stone, a blue stone, or nothing, can you remove stones from the board such that each column is monochromatic, and no row is empty?

Given a nondeterministic finite-state automoton (NFA) M, is there an integer N that will be rejected by M.

Given twon NFA's do they recognize different finite languages?

E3SAT-WITH-MAJORITY: is there a satisfying assignment in which more than half of the clauses have all 3 variables satisfied?

In the game 'Shanghai', tiles with images are placed face-up into stacks of various shapes. The goal is to remove pairs of matching tiles until all of the tiles are removed, but tiles can only be removed from the top of a stack. This can be shown NP-hard even when all of the tile positions are known by reducing from 3SAT.

[**?**] Battleships can be shown NP-complete by a reduction from 3SAT-3,4 (3SAT where each variable occurs either 3 or 4 times).

[**?**] The game Trackmania is NP-complete by a reduction from 3SAT.

References

[1] Erik Demaine, William Gasarch, and Mohammad Hajiaghayi *Fun With Hardness: Algorithmic Lower Bounds* .

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[3] Merlijn Sevenster *BATTLESHIPS AS DECISION PROBLEM* September 2004

[4] Franck Dernoncourt *TrackMania is NP-Complete* November 2014