CMSC 858M: Algorithmic Lower Bounds
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Chapter 3: NP-Hardness Via SAT and Planar SAT

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1 General Overview

This is a really interesting chapter. The material is presented both well and (mostly) consistently formatted, which makes it seem polished. My main overall suggestion would be wherever possible, provide some intuition about how one might actually devise the gadgets used in proofs. In many cases it appears that they came out of nowhere, and while these lead to correct proofs, they are not the most satisfying for a reader.

2 Section 3.1: Introduction

Good – no substantial comments. I like that you acknowledge up front that PL-3SAT is not a natural problem, but provide motivation for looking at it. In particular, I found the second argument interesting.

3 Section 3.2: Clique, Independent Set, and Vertex Cover

I think the main proof of this section is presented very nicely, and the corresponding image (Figure 3.1) is very helpful.

4 Section 3.3: PL-3SAT

One quick bug I just noticed that I did not put in the bug report I sent is that I think in the paragraph above Figure 3.2, the colors are backwards. It appears there is a red edge between x and C if $\neg x$ is in C.

I don't know if this is possible, but I think it would be helpful to provide some intuition for how one might arrive at a gadget like Figure 3.3. When reading the proof, of Theorem 3.2.2., I get that one can verify the gadget works, but it feels like it comes out of nowhere, which makes the proof seem less satisfying than it possibly could be.

5 Section 3.4: Graph Coloring and Variants

I think the presentation of the gadget in proof of Theorem 3.4.1. is done much better here than the presentation in previous section. It's easy to follow. However, I strongly dislike the wall of images from pages 74 to top of 77; they disrupt the flow of the chapter.

Two quick things / bugs that I didn't notice on first pass concerning (1) on page 77. I believe "Brook's Theorem" should be "Brooks' theorem". Also, there is a slight caveat from Brooks' theorem that is left out. It states such graphs are always 3-colorable **unless** they are a complete graph or an odd cycle. I'm assuming one could check whether those two cases occur in polytime, but I think this should be mentioned.

6 Sections 3.5/3.6: Vertex Cover and Variants/ Dominating Set

Good – no real comments. By this time in the chapter, these proofs start to feel routine (I mean that in a good way).

7 Section 3.7: Planar Directed Hamiltonian Graphs and Variants

I think the presentation of the proof sketch of Theorem 3.7.2. could definitely be improved. In particular, I got thrown off by the introduction of "alternating-wire gadgets" without sufficient explanation of what these are / where this term is coming from. However, the corresponding figure was helpful.

8 Section 3.8: Shakashaka

I like the idea of putting a fun application here that pertains to PL-3SAT, but as I mentioned in the bug report, there are missing figures (one seems to be in the next section?) and the proof seems incomplete, so I can't really follow what's going on in this section. Also extremely minor, but (another thing I forgot on bug report) missing a space after "Erik Demaine et. al" and before the citation.

9 Section 3.9: Planar Rectilinear 3SAT and Planar Monotone Rectilinear 3SAT

Another thing I forgot to pick up in first pass through chapter: there are multiple occurrences of "Rectilinear" and of "Rectlinear" (no middle "i"), and this problem occurs even in the section title. I'm assuming these should be one consistent spelling.

In general, the presentation of this section is fine, but I don't think this is a terribly interesting or natural problem, and the chapter seems to have more than enough examples / applications already. If you decide to keep it, I'm glad you at least mention the application to METAFONT Labelling to show that this problem is not completely out of nowhere.

10 Section 3.10: Planar 1-in-3SAT

Good – no real comments, although I supposed it's worth saying the labels in the corresponding figure (currently Figure 3.15) are a bit difficult to read without zooming in.

11 Section 3.11 - 3.14

Good – no real comments. These sections read pretty routinely. Note the section title of 3.13 is misspelled.

12 Section 3.15: Flattening Fixed-Angle Chains

This is an interesting "real-life" example that I like. Also change "NP-Hard" to "NP-hard" in proof of Theorem 3.15.2. to stay consistent in presentation.

13 Important Related Problems

In the spirit of the chapter, here are some problems I found that use reductions from planar 3SAT (or a close variant).

- Planar k-means is an important clustering problem. Formally, the question is: *Given a finite set* $S = \{p_1, p_2, \ldots, p_n\}$ *of points with rational coordinates in* $\mathbb{R}^2$*, an integer* $k \geq 1$*, and a bound* $R \in \mathbb{R}$ *which is a rational number, determine if there exists* k *centers* $\{c_1, \ldots, c_k\}$ *in* $\mathbb{R}^2$ *such* *that* $\sum_{i=1}^{n}\left(\min_{1\leq j\leq k}[d(p_i,c_j)]^2\right)\leq R$. Mahajan et al. [6] determined this problem is NP-hard using a reduction from planar 3SAT.

- One problem in robotics planning is multi-robot path planning problems on planar graphs. Essentially, given a planar graph, robots, and destinations, find a set of paths along the graph such that no two paths lead to a collision. A problem of interest, MTTPMPP, asks whether there is a solution path set with a total arrival time no more than some integer. Yu [10] showed this problem is NP-hard using a reduction from monotone planar 3SAT.

- Another important robotics problem deals with designing a process to bring together parts of a product into their final state. This problem motivates the assembly partitioning problem: *Given a set of parts in their final placement in a product, partition them into two sets, such that they can be moved sufficiently far away from each other without collisions.* Agarwal et al. [1] showed that even with a simplified version, this problem is NP-complete by a reduction from planar 3SAT.

- The following problem has connections to graphs with zero-sum flow. *Given a planar* $(3,4)$*-semiregular graph* G*, determine whether there exists a vector with entries belonging to* $\{\pm 1, \pm 2\}$ *in the null space of the 0-1 incidence matrix of* G*.* Using a reduction from a variant of Planar 1-in-3 SAT, Dehghan et al. [3] showed this problem to be NP-complete.

- The notion of a planar CNF can be extended to the notion of a *strongly planar CNF*. We say a CNF is strongly planar if the corresponding graph in the normal planar CNF case is still planar when adding edges between any two literals x and $\neg x$ of the same variable x. Like its non-strong version, strongly planar 3SAT is NP-complete, as well as strongly planar 1-in-3SAT and strongly planar NAE 3SAT [9].

- The following is the tracking paths problem. *Given a graph with a source* s *and destination* t*, find the smallest subset of vertices whose intersection with any* $s-t$ *path results in a unique sequence.* Eppstein et al. [4] proved this problem is NP-complete even in the case of planar graphs using a reduction from planar 3SAT.

- Alexander Pilz [7] investigated the complexity of a variant of planar 3-SAT. He showed that if you assume the formula's corresponding graph is augmented by the edges of a Hamiltonian cycle that first passes through all variables and then through all clauses in a way that the resulting graph is still planar, determining the satisfiability of a 3-SAT formula remains NP-complete.

- minSAT is the problem of finding the boolean assignments that satisfy the minimum number of given disjunctive clauses. In the case of planar min2SAT, Guibas et al. [5] proved the problem NP-hard using a reduction from planar 3SAT.

- In his thesis, Tippenhauer [8] discusses a myriad of variants of planar 3-SAT. He covers the classic variants from the text, but extends into very specific versions, such as when the number of variables are bounded and the formulas are monotone.
- A lucky labeling of some graph G is a mapping ℓ from G's vertices to the natural numbers such that for every two adjacent vertices u and v, $\sum_{(w,v)} \ell(w) \neq \sum_{(w,u)} \ell(w)$. Ahadi et al. [2] used a reduction from planar 3SAT to show it is NP-complete to determine whether the smallest range such that a given planar 3-colorable graph has a lucky labeling is 2.

References

[1] Pankaj K Agarwal, Boris Aronov, Tzvika Geft, and Dan Halperin. On two-handed planar assembly partitioning. *arXiv preprint arXiv:2009.12369*, 2020.

[2] Arash Ahadi, Ali Dehghan, M Kazemi, and E Mollaahmadi. Computation of lucky number of planar graphs is NP-hard. *Information processing letters*, 112(4):109–112, 2012.

[3] Ali Dehghan, Mohammad-Reza Sadeghi, and Arash Ahadi. Not-all-equal and 1-in-degree decompositions: algorithmic complexity and applications. *Algorithmica*, 80(12):3704–3727, 2018.

[4] David Eppstein, Michael T Goodrich, James A Liu, and Pedro Matias. Tracking paths in planar graphs. *arXiv preprint arXiv:1908.05445*, 2019.

[5] Leonidas J Guibas, John E Hershberger, Joseph SB Mitchell, and Jack Scott Snoeyink. Approximating polygons and subdivisions with minimum-link paths. *International Journal of Computational Geometry & Applications*, 3(04):383–415, 1993.

[6] Meena Mahajan, Prajakta Nimbhorkar, and Kasturi Varadarajan. The planar k-means problem is NP-hard. In *International Workshop on Algorithms and Computation*, pages 274–285. Springer, 2009.

[7] Alexander Pilz. Planar 3-SAT with a clause/variable cycle. *arXiv preprint arXiv:1710.07476*, 2017.

[8] Simon Tippenhauer and Wolfgang Muzler. On planar 3-SAT and its variants. *Fachbereich Mathematik und Informatik der Freien Universitat Berlin*, 2016.

[9] Lidong Wu. On strongly planar 3SAT. *Journal of Combinatorial Optimization*, 32(1):293–298, 2016.

[10] Jingjin Yu. Intractability of optimal multirobot path planning on planar graphs. *IEEE Robotics and Automation Letters*, 1(1):33–40, 2015.