Hardness of Approximation: 10 Related Problems

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We survey several problems related to approximation algorithms and the hardness of approximation which are not currently in the book. There are personal biases in these problem choices since they are problems which the author has read about or worked on, but they should be interesting nonetheless. In particular, they will overly represent facility location-type problems. We also include some other topics related to hardness of approximation, which Bill and Mohammad can choose to add to the chapters if they think they are interesting. Throughout the sections, we will also mention what directions are open with regards to the mentioned problems.

1 Inapproximability

In the current sections on hardness of approximation, several classes of problems are identified: APX-Complete, LOG-APX-Complete, POLY-APX-Complete, and EXP-APX-Complete. One class of problems which weren't mentioned were the inapproximable problems (I don't know if there's an official name). That is, the book doesn't mention those problems where it is NP-Hard to obtain a non-infinite approximation factor. Usually, this is obtained when it is NP-Complete to determine whether or not the optimal solution is exactly 0.

bProblem 1: (**Scheduling Jobs With Deadlines** [1]) We are given n jobs which need to be scheduled on a single machine, which can process one job at a time. For each job j, the job takes time p_j, may begin no earlier than the release time r_j, and is due at time d_j. Suppose we finish processing job j at time C_j; then the lateness is defined as $L_j = C_j - d_j$. We want to schedule the n jobs such that the maximum lateness $L_{max} = \max_{j \in [n]} L_j$ is minimized.

- Deciding if there is a schedule such that $L_{max} \leq 0$ is NP-Hard (i.e., no approximation algorithm exists)
- If we assume all due dates are negative, there exists a 2-approximation algorithm.

Problem 2: (**Dynamic k-Supplier** [2]) We are given a metric space (X, d). An instance of Dynamic k-Supplier consists of $T \geq 2$ timesteps, a set of clients $C_t \subseteq X$ for time $t \in [T]$, a set of candidate facility locations $F_t \subseteq X$ for time $t \in [T]$, a budget on the number of facilities to place $k \in N$, and a budget on the movement cost B. We are asked to compute a sequence of multi-sets of facilities

$\{A_t\}_{t=1}^T$, with $A_t \subseteq F_t$ and $|A_t| = k$, minimizing the maximum service cost of any client $\max_{t\in[T]} \max_{j\in C_t} d(j, A_t)$ subject to the constraint that there must exist a perfect matching between A_t and A_{t+1} (for all t) such that the distance between each matched pair is at most B.

- For $T = 2$, there exists a 3-approximation algorithm and this is best possible unless P=NP.
- For $T \geq 3$, there can be no approximation algorithm for this problem unless P=NP.

Problem 3a: (**γ-Colorful k-Center** [3]) We are given a metric space (X, d), γ colors, and a budget k. We are also given labels $X_\ell \subseteq X$ and covering requirement $m_\ell \in N \cup \{0\}$ for each $\ell \in [\gamma]$. We want to find the smallest radius $r \in R_{\geq 0}$ together with centers $C \subseteq X$ such that $|C| \leq k$ and $|B(C, r) \cap X_\ell| \geq m_\ell$ for each $\ell \in [\gamma]$, where $B(C, r)$ is the open ball of radius R centered at elements in C.

- When γ is viewed as a constant, there exists constant factor approximation algorithms.
- When γ is viewed as input, there can be no approximation algorithm for the problem unless P=NP.

In addition to the ones here, [11] also gives a problem which is polynomial-time inapproximable. This was found after I had written 10 problems already, so I have not included it in the writeup.

2 Bicriteria Inapproximability

Often, when there are strong hardness results or technical difficulties in developing approximation algorithms, one may turn to bicriteria approximation algorithms. For an optimization with a constraint k (think of this as a budget) and an objective, an (α, β)-approximation algorithm is one which outputs a solution with objective at most α times optimal and violates the budget k by at most β. One can also obtain hardness of approximation results of this form: even if we violate the objective by a factor of β, there cannot exist any algorithm which outputs a solution within an α factor of the optimal solution. We mention some examples here which the authors have seen, including one problem for which bicriteria approximation algorithms are the only constant factor approximation algorithms known for the problem:

Problem 3b: (**γ-Colorful k-Center** [3]) We are given a metric space (X, d), γ colors, and a budget k. We are also given labels $X_\ell \subseteq X$ and covering requirement $m_\ell \in N \cup \{0\}$ for each $\ell \in [\gamma]$. We want to find the smallest radius $r \in R_{\geq 0}$ together with centers $C \subseteq X$ such that $|C| \leq k$ and $|B(C, r) \cap X_\ell| \geq m_\ell$ for each $\ell \in [\gamma]$, where $B(C, r)$ is the open ball of radius R centered at elements in C.

- When γ is viewed as input, there can be no approximation algorithm for the problem even when violating the budget constraint by $O(\log n)$, unless P=NP.

- There exists a $(1, O(\log \gamma))$-bicriteria approximation algorithm for the problem.

Problem 4: (**Capacitated k-Center** [4]) We are given a metric space (X, d), a budget k, and a capacity constraint L. We want to find a subset $S \subseteq X$ of size at most k such that $\max_{i \in X} \min_{j \in S} d(i, j)$ is minimized subject to the constraint at most L clients are assigned to each center in S.

- A 6-approximation algorithm is known for the problem.

- Let $c = \frac{x+1}{x}$ for any $x \geq 1$. When violating budget by $\frac{2}{c}$ and capacity by c, we can achieve a 2-approximation algorithm.

- No $2 - \epsilon$ approximation algorithm exists, even when violating the budget and capacity constraints by any constant factor.

Problem 5: (**Capacitated k-Median** [5]) We are given a metric space (X, d), a budget k, and a capacity constraint L. We want to find a subset $S \subseteq X$ of size at most k such that $\sum_{i \in X} \min_{j \in S} d(i, j)$ is minimized subject to the constraint at most L clients are assigned to each center in S.

- The problem is NP-Hard.

- There exists $(O\left(\frac{1}{\epsilon^2} \log(1/\epsilon)\right), 1 + \epsilon)$-bicriteria approximation algorithms for the problem.

- It is open whether or not there is a non-bicriteria constant factor approximation algorithm.

3 Open Problems

Problem 6: (**Spectral Radius Minimization** [6]) We are given an undirected graph $G = (V, E)$ and a target T. We want to pick a subset $S \subseteq$ of vertices such that the spectral radius (defined as the maximum eigenvalue of the adjacency matrix) is at most T and the number of remaining vertices $|V - S|$ is maximized.

- It is known that the problem is NP-Hard.

- There exists $O(\log n)$-approximation algorithms for the problem based on semi-definite programming.

- Whether or not a polynomial-time approximation scheme exists is open (i.e., is it APX-Hard?).

Problem 7: (Minimum Size Bounded-Capacity Cut [7]) We are given an undirected graph $G = (V, E)$, edge capacities c_e, a budget k, a source vertex s, and a sink vertex t. We want to find an $s - t$ cut (S, S^c) with $s \in S$ of capacity at most k such that $|S|$ is minimized.

- **The problem is NP-Hard.**
- **There exists a $(\frac{1}{\lambda}, \frac{1}{1-\lambda})$-bicriteria approximation algorithm, for any $\lambda \in (0, 1)$.**
- **To the best of my knowledge, it is not known whether it is APX-Hard.**

4 Other Problems

In this final section, we will state some problems related to hardness of approximation which we have reduced from or for which the proof was quite cool/elegant. The final problem can be included as an example in the POLY-APX-Complete section of the book.

Problem 8: (Densest k-Subgraph [8]) We are given a graph $G = (V, E)$ and an integer $k \in N$. For a subgraph $S = (E_S, V_S)$ of G, define the density to be $d(S) = \frac{|E_S|}{|V_S|}$. We want to find a subgraph $S = (E_S, V_S)$ with $|V_S| = k$ such that the density $d(S)$ is maximized.

- **For all $\epsilon > 0$, there exists an $n^{1/4+\epsilon}$-approximation algorithm.**
- **Unless the exponential time hypothesis fails, there is no $n^{1/(\log\log\log n)}g(n)$-approximation algorithm for the problem..**
- **Unless $NP \subseteq \bigcap_{\epsilon>0} BPTIME(2^{n^\epsilon})$, no polynomial-time approximation scheme exists for the problem.**

Problem 9: Bin-packing Problem [9]) We are given n items with sizes $1 > a_1 \geq a_2 \geq \ldots \geq a_n > 0$. We wish to pack the items into the fewest bins possible, where each bin can hold any subset of pieces of total size at most 1.

- **There can not exist an α-approximation algorithm for $\alpha < \frac{3}{2}$.**
- **There exists an algorithm which outputs a solution using $OPT+1$ bins (this is already in the book).**

Problem 10: Edge-Disjoint Paths Problem [10]) We are given a directed graph $G = (V, A)$ and k source-sink pairs $s_i, t_i \in V$. The goal of the problem is to find edge-disjoint paths so that the number of source-sink which have a path from s_i to t_i is maximized.

- **A simple greedy algorithm obtains an $\Omega(1/\ell)$-approximation algorithm for the problem, where $\ell = \max\{\sqrt{m}, diam(G)\}$.**
- **For any $\epsilon > 0$, there is no $\Omega(m^{-1/2+\epsilon})$-approximation algorithm for the problem unless $P = NP$.**

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