High School Proofs for Better Bounds on the Quadratic van der Waerden Numbers

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Abstract

A corollary of the polynomial van der Waerden theorem is that, for any polynomial \( p(x) \in \mathbb{Z}[x] \) with constant term 0, for any \( c \in \mathbb{N} \), there exists \( W \in \mathbb{N} \) such that, for all \( c \)-colorings of \( \{1, \ldots, W\} \) there exists \( a, d \) such that \( a \) and \( a + p(d) \) are the same color. The proof of the polynomial van der Waerden theorem, and even of these corollaries, is difficult and gives enormous upper bounds for \( W \). We consider just quadratic polynomials. For \( c = 2, 3 \) we obtain reasonable bounds, and for \( c = 4 \) for some quadratics we obtain reasonable bounds, using simple methods.

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1. Introduction

We use the following standard definition.

\textbf{Definition 1.1.} Let \( \mathbb{Z} \) be the set of integers, \( \mathbb{N} \) be the set of non-negative integers, and \( \mathbb{N}^+ \) be the set of positive integers. Let \( [W] \) be the set \( \{1, \ldots, W\} \) (where \( W \in \mathbb{N} \)).
In this paper we will give *High School Proofs (HS Proof)* of theorems. The term *High School Proof* is not a formal term. We use it to mean a proof that can be explained to a bright high school student. We use the term *High School Proof* since (1) the terms *elementary* is ambiguous, and (2) the term *Combinatorial* is not quite right since (a) the rather difficult proof of Szemerédi’s Theorem is combinatorial, and (b) the rather difficult proof of Gower’s bound is mostly combinatorial.

Recall van der Waerden’s Theorem \[1, 2\] (see also the books by Graham-Rothchild-Spencer \[3\] and Landman-Robertson \[4\]).

**Theorem 1.2.** For any \(k \in \mathbb{N}\), for any \(c \in \mathbb{N}\), there exists \(W = W(k, c)\), such that for any \(c\)-coloring of \([W]\), there exists \(a, d \in \mathbb{N}\), \(d \neq 0\), such that \(a, a + d, \ldots, a + (k - 1)d\) are all the same color.

The original proof by van der Waerden was HS but yielded bounds on \(W\) that were not primitive recursive \[3\]. Shelah \[5\] gave a HS proof that yielded primitive recursive bounds on \(W\). These bounds were still quite large in that they really cannot be written down nicely. Gowers \[6\] gave a non-HS proof that yielded bounds that can be written down:

\[W(k, c) \leq 2^{2^{2^k + 9}}\]

We discuss a known generalization of van der Waerden’s theorem. Note that the conclusion of van der Waerden’s theorem is that

\[a, a + d, a + 2d, \ldots, a + (k - 1)d\] are the same color.

Can we replace \(d, 2d, \ldots, (k - 1)d\) by other functions of \(d\)? Yes. We can replace them with polynomials in \(\mathbb{Z}[x]\) that have no constant term. Here is the Polynomial van der Waerden Theorem:

**Theorem 1.3.** Let \(p_1, \ldots, p_k \in \mathbb{Z}[x]\) such that, for \(1 \leq i \leq k\), \(p_i(0) = 0\). Let \(c \in \mathbb{N}\). Then there exists \(W = W(p_1, \ldots, p_k; c)\) such that, for any \(c\)-coloring of
There exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a + p_1(d), \ldots, a + p_k(d)$ are all the same color.

For $k = 1$, this theorem was proven independently by Furstenberg [7] and Sárközy [8]. Bergelson and Leibman [9] proved the general result using ergodic methods (not a HS proof). These proofs yielded no upper bounds on $W(p_1, \ldots, p_k; c)$. Walters [10] obtained a HS proof of Theorem 1.3 but the bounds on $W$ were not primitive recursive. Shelah [11] gave a (non HS) proof that yielded primitive recursive bounds on $W$. These bounds were still quite large in that they really cannot be written down nicely. Nobody has obtained a proof that yields bounds one can write down.

Peluse [12] and Peluse and Prediville [13] proved density results that can be translated into bounds for some polynomial van der Waerden numbers.

1. Peluse and Predville [13] showed that there exists a $d$ such that for large $n$, $W(x, x^2; (\log \log n)^d) \leq n$.
2. Peluse [12] showed that if $p_1, \ldots, p_m \in \mathbb{Z}[x]$ are polynomials of different degrees then there exists a constant $d$ (which depends on $p_1, \ldots, p_m$) such that, for large $n$, $W(p_1(x), \ldots, p_m(x); (\log \log n)^d) \leq n$.
3. Peluse and Predville [14] showed that there exists a $d$ such that for large $n$, $W(x, x^2; (\log n)^d) \leq n$.

These proofs are not HS.

We are interested in the case of $W(ax^2 + bx; c)$ where $c = 2, 3, 4$. Furstenberg’s proof showed that $W(x^2; c)$ exists; however, his proof gave no upper bounds. Sárközy’s proof showed that $W(x^2; c) \leq 2^{O(c^3)}$. Pintz, Steiger, and Szemerédi [15] (see also [16] for exposition) showed that $W(x^2; c) \leq 2^{O(c^{0.0001})}$. The 0.0001 can be replaced with any smaller constant; however, in that case the constant associated with the big-O will increase. It is possible that either Sárközy’s proof of $W(x^2; c) \leq 2^{O(c^3)}$ or Pintz, Steiger, and Szemerédi proof of $W(x^2; c) \leq 2^{O(c^{0.0001})}$ could be modified with a fixed value of $c$ such as 4. That may lead to an improvement on our bound on $W(x^2; 4)$; however, such a proof would not be HS.
Harnel, Lyall, and Rice [17] showed that there exists a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ such that

$$W(ax^2 + bx; c) \leq 2f(a,b)c^{0.0001}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function $f$ will be bigger).

Later Rice [16] showed that, for all $k$, there exists a function $f : \mathbb{Z}^k \to \mathbb{N}$ such that

$$W(a_k x^k + \cdots + a_1 x; c) \leq 2f(a_k, \ldots, a_1)c^{0.0001}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function $f$ will be bigger). Rice (personal communication) later obtained the following more precise result: for all $\epsilon > 0$, for all $a_1, \ldots, a_k \in \mathbb{Z}$, for $J = |a_1| + \cdots + |a_k|:

$$W(a_k x^k + \cdots + a_1 x; c) \leq 2^{2^{2^{100k^2/\epsilon}}} + 2^{2^{(100k^4 \log J)^{100}}} + 2^{e^\epsilon}$$

In summary, the known bounds on $W(ax^2 + bx; c)$ are large.

In this paper we show that, for some $p \in \mathbb{Z}[x]$ and $c = 2, 3, 4$, one can obtain much better bounds on $W(p(x); c)$. Our proofs will be purely combinatorial and much easier than those of Walters, Shelah, and Peluse. We hasten to point out that they proved the full polynomial van der Waerden theorem whereas we only prove it in very special cases.

We will show the following.

- For all $a \in \mathbb{Z}$, $W(ax; c) = |ac| + 1$.
- For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \leq 12|a| + 6|b|$. We actually obtain more precise bounds than that depending on how $a, b$ are related to each other.
- In Appendix A is a table of some exact values of $W(ax^2 + bx; 2)$.
- For all $a \in \mathbb{N}$, $a \geq 1$, $W(ax^2 + (a - 1)x; 2) = 8a - 3$. 


• \(W(x^2; 3) = 29\) and, for all \(a \in \mathbb{Z}\), \(W(ax^2; 3) = 28a + 1\).

• For \(a, b \in \mathbb{Z}\), \(W(ax^2 + bx; 3) \leq O(a^2b^5)\). In Appendix B is a table of some exact values of \(W(ax^2 + bx; 3)\).

• \(W(x^2; 4) \leq 84, 149, 474, 894, 213, 522\). In Appendix C is a table of some upper bounds on \(W(ax^2 + bx; 4)\).

2. Preliminaries

Definition 2.1. Let \(c \in \mathbb{N}^+\) and \(W \in \mathbb{N}^+\).

1. A \(c\)-coloring of \([W]\) is a mapping \([W] \to [c]\).

2. Let \(p \in \mathbb{Z}[x]\). A \((p; c)\)-proper coloring of \([W]\) is a \(c\)-coloring of \([W]\) such that, for all \(x, y \in [W]\), if \(y - x = p(d)\) for some \(d \in \mathbb{N}^+\), then \(x\) and \(y\) have different colors. When the context is clear, we will often write proper \(c\)-coloring or simply proper coloring.

Note that the polynomial van der Waerden number, \(W = W(p(x); c)\), is the least number such that there is no \((p; c)\)-proper coloring of \([W]\).

Although we care about proper \((p; c)\)-colorings, we need a more general notion:

Definition 2.2. Let \(F \subseteq \mathbb{Z}\), \(c \in \mathbb{N}^+\), and \(W \in \mathbb{N}^+\).

• An \((F; c)\)-proper coloring of \([W]\) is a \(c\)-coloring of \([W]\) such that, for all \(x, y \in [W]\) with \(y - x \in F\), \(x\) and \(y\) have different colors.

• \(W = W(F; c)\) is the least number such that there is no \((F; c)\)-proper coloring of \([W]\). If no such number exists, we set \(W(F; c) = \infty\).

• In the above definitions \(F\) is the set of forbidden distances. We will use this term for polynomial van der Waerden numbers as well. For example, if looking at \(W(3x^2; c)\) the forbidden distances are \(3 \times 1^2, 3 \times 2^2, \ldots\)

We leave the following easy lemma to the reader.
Lemma 2.3. Let $c \in \mathbb{N}^+$.  

1. If $0 \in F$ then $W(F; c) = 1$.
2. Assume $f \in F$. Let $F' = F \cup \{-f\}$. Then $W(F; c) = W(F'; c)$.

Lemma 2.4. Let $p \in \mathbb{Z}[x]$, $a \in \mathbb{Z}$, and $c \in \mathbb{N}$. Then

$$W(ap; c) = a(W(p; c) - 1) + 1.$$ 

Proof:

1) $W(ap; c) \leq a(W(p; c) - 1) + 1$:

Assume, by way of contradiction, that $W(ap; c) \geq a(W(p; c) - 1) + 2$. Hence there exists COL, an $(ap; c)$-proper coloring of $[a(W(p; c) - 1) + 1]$. Note that, for all $x$, $ap(x)$ is a forbidden distance for COL.

We use COL to define COL', a proper $(p; c)$-coloring of $[W(p; c)]$; which contradicts the definition of $W(p; c)$.

For $1 \leq i \leq W(p; c)$ let

$$COL'(i) = COL(a(i - 1) + 1).$$

Note that

$$COL'(j) = COL'(i) \text{ iff } COL(a(j - 1) + 1) = COL(a(i - 1) + 1).$$

Suppose $j - i$ is a forbidden distance for COL'. Then there exists $x$ such that $j - i = p(x)$. Then

$$a(j - 1) + 1 - (a(i - 1) + 1) = a(j - i) = ap(x), \text{ a forbidden distance for COL.}$$

Hence $COL(a(j - 1) + 1) \neq COL(a(i - 1) + 1)$, so $COL'(j) \neq COL'(i)$. Therefore COL' is a proper $(p; c)$-coloring of $[W(p; c)]$.

2) $W(ap; c) \geq a(W(p; c) - 1) + 1$:

To show $W(ap; c) \geq W(ap; c) \geq a(W(p; c) - 1) + 1$ we need to give a proper $(ap; c)$-coloring of $[a(W(p; c) - 1)]$. 

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Let \( X \) be a number to be named later. Let \( \text{COL}' \) be a proper \((p; c)\)-coloring of \([X]\). The reader can easily verify that \( \text{COL} \), defined below, is a proper \((ap; c)\)-coloring of \([aX]\).

- Color \( 1, \ldots, a \) with \( \text{COL}'(1) \).
- Color \( a + 1, \ldots, 2a \) with \( \text{COL}'(2) \).
- \( \vdots \)
- Color \( (X - 1)a + 1, \ldots, Xa \) with \( \text{COL}'(X) \).

Take \( X = W(p; c) - 1 \). By definition there exists \( \text{COL}' \), a proper \((p; c)\)-coloring of \([X]\). Hence \( \text{COL} \) is a proper \((ap; c)\)-coloring of \([aX] = [a(W(p; c) - 1)]\) which is what we need.

3. The Exact Value of \( W(ax; 2) \)

For completeness we cover linear polynomials, for which we obtain a complete solution.

**Theorem 3.1.** Let \( a \in \mathbb{Z} \) and \( c \in \mathbb{N}^+ \). Then

\[
W(ax; c) = |ac| + 1 .
\]

**Proof:** By Lemma 2.3.1 we have the case of \( a = 0 \). By Lemma 2.3.2 we can assume that \( |a| \) is a forbidden distance.

\[
W(ax; c) \leq |ac| + 1:
\]

By setting \( x = 1, 2, \ldots, c \) we get forbidden distances \( |a|, |2a|, \ldots, |ca| \). So \( 1, |a| + 1, |2a| + 1, \ldots, |ca| + 1 \) must all be different colors, but there are only \( c \) colors.

\[
W(ax; c) \geq |ac| + 1:
\]

We can properly \( c \)-color \([ca]\):

- Color \( 1, \ldots, |a| \) with 1.
- Color \( |a| + 1, \ldots, |2a| \) with 2.
4. Upper Bounds on $W(ax^2 + bx; 2)$

**Theorem 4.1.** Let $a, b \in \mathbb{N}$ with $a \geq 1$ and $b \geq 0$.

1. $W(ax^2 + bx, 2) \leq 12a + 6b + 1$.
2. If $b \geq 3a$ then $W(-ax^2 + bx, 2) \leq 6b - 12a + 1$.
3. If $2a \leq b \leq 3a$ then $W(-ax^2 + bx, 2) \leq 3b - 3a + 1$.
4. If $0 \leq b \leq a$ then $W(-ax^2 + bx, 2) \leq 12a - 6b + 1$.

**Proof:** Let $d$ be a forbidden distance. For all $y$ such that, $y, y+d, y+2d, y+3d$ are colored

$$
\text{COL}(y) = R \rightarrow \text{COL}(y + d) = B \rightarrow \text{COL}(y + 2d) = R \rightarrow \text{COL}(y + 3d) = B.
$$

Hence $3d$ is also a forbidden distance.

1) $W(ax^2 + bx; 2)$. By plugging in $x = 1, 2, 3$ we find forbidden distances:

$$
\{a + b, 4a + 2b, 9a + 3b\}.
$$

We will use the following forbidden distances:

$$
\{3a + 3b, 3(4a + 2b), 9a + 3b\} = \{3a + 3b, 12a + 6b, 9a + 3b\}.
$$

Assume, by way of contradiction that there is a proper $W(x^2; 2)$-coloring of $[12a + 6b + 1]$. We can assume that $\text{COL}(1) = R$. Note that
COL(1) = \( R \rightarrow COL(1 + (3a + 3b)) = B \rightarrow COL(1 + (3a + 3b) + (9a + 3b)) = R \).

We simplify to obtain \( COL(12a + 6b + 1) = R \).

\[ COL(12a + 6b + 1) = R \rightarrow COL(12a + 6b + 1 - (12a + 6b)) = B. \]

We simplify to obtain \( COL(1) = B \) which is a contradiction.

The key to the last proof was that

- \((3a + 3b) + (9a + 3b) - (12a + 6b) = 0.\)
- \(COL\) is defined on \((3a + 3b) + (9a + 3b) = 12a + 6b.\)

For all later proofs we just give positive forbidden distances \(d_1, d_2, d_3\) such that \(d_1 + d_2 - d_3 = 0\), and conclude that the bound is \(d_1 + d_2 + 1\). We abbreviate Forbidden Distances by FD.

We now consider \(W(-ax^2 + bx; 2)\):

2) \(b \geq 3a\). FD: \(\{3b - 3a, 6b - 12a, 3b - 9a\}\). \((3b - 3a) + (3b - 9a) - (6b - 12a) = 0.\)

3) \(2a \leq b \leq 3a\). FD: \(\{3b - 3a, 6b - 12a, 9a - 3b\}\). \((6b - 12a) + (9a - 3b) - (3b - 3a) = 0.\)

4) \(a \leq b \leq 2a\). FD: \(\{3b - 3a, 12a - 6b, 9a - 3b\}\). \((3b - 3a) + (12a - 6b) - (9a - 3b) = 0.\)

5) \(0 \leq b \leq a\). FD: \(\{3a - 3b, 12a - 6b, 9a - 3b\}\). \((3a - 3b) + (9a - 3b) - (12a - 6b) = 0.\)

**Corollary 4.2.** For all \(a, b \in \mathbb{Z}\), \(W(ax^2 + bx; 2) \leq 12|a| + 6|b|\).

The bounds on \(W(ax^2 + bx; 2)\) (and the others) from Theorem 4.1 hold for all \(a, b\); however, for particular \(a, b\) better bounds can often be found. We give a class of examples.
Theorem 4.3. Let \( a \in \mathbb{N} \) with \( a \geq 1 \). Then \( W(ax^2 + (a - 1)x; 2) = 8a - 3 \).

Proof:
1) \( W(ax^2 + (a - 1)x; 2) \leq 8a - 3 \).

Let \( \text{COL} : [8a - 3] \to \{R, B\} \).

By plugging in \( x = 1, 2 \) we find forbidden distances: \( \{2a - 1, 6a - 2\} \). Since \( 2a - 1 \) is a forbidden distance, so is \( 3(2a - 1) = 6a - 3 \). We will use forbidden distances \( \{6a - 3, 6a - 2\} \).

Let \( y \leq 2a - 1 \). Assume \( \text{COL}(y) = R \). Then

\[
\text{COL}(y) = R \rightarrow \text{COL}(y + (6a - 2)) = B \rightarrow \text{COL}(y + (6a - 2) - (6a - 3)) = R.
\]

To simplify, \( \text{COL}(y + 1) = R \). We needed \( y \leq 2a - 1 \) since we needed \( y + (6a - 2) \leq 8a - 3 \) so that \( y + (6a - 2) \) is in the domain of \( \text{COL} \).

Assume \( \text{COL}(1) = R \). Then by applying the above we get \( \text{COL}(2) = R, \ldots \), \( \text{COL}(2a) = R \). However, since \( \text{COL}(1) = R \) and \( 2a - 1 \) is a forbidden distance, \( \text{COL}(2a) = B \). This is a contradiction.

2) \( W(ax^2 + (a - 1)x; 2) \geq 8a - 3 \).

We give a coloring \( \text{COL} \) of \([8a - 4]\) such that, for all \( x, y \in [8a - 4] \) with \( |x - y| \in \{2a - 1, 6a - 2\} \), \( \text{COL}(x) \neq \text{COL}(y) \). All other forbidden distances are larger than \( 8a - 4 \) and hence irrelevant.

Here is the coloring:

1. For \( 1 \leq y \leq 2a - 1 \), \( \text{COL}(y) = R \).
2. For \( 2a \leq y \leq 4a - 2 \), \( \text{COL}(y) = B \).
3. For \( 4a - 1 \leq y \leq 6a - 3 \), \( \text{COL}(y) = R \).
4. For \( 6a - 2 \leq y \leq 8a - 4 \), \( \text{COL}(y) = B \).

The reader can verify that this coloring suffices. \( \blacksquare \)

In Appendix A is a table of some exact values of \( W(ax^2 + bx; 2) \).
5. \( W(ax^2; 3) = 28a + 1 \)

In this section we will show that \( W(x^2; 3) = 29 \) and then \( W(ax^2; 3) = 28a + 1 \).

We first show a weaker theorem which will be a good warm-up to our work on 4-colorings in Section 7.

**Theorem 5.1.** \( W(x^2; 3) \leq 1 + 41^2 = 1682. \)

**Proof:**

Figure 1: In any \((x^2, 3)\)-proper coloring, \( \text{COL}(x) = \text{COL}(x + 41) \)

Assume, by way of contradiction, that \( \text{COL} \) is an \((x^2; 3)\)-proper coloring of \([1 + 41^2]\). Figure 1 shows some constraints on \( \text{COL} \): \( \text{COL} \) restricted to \( \{1, 17, 26, 42\} \) has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

We can assume \( \text{COL}(1) = R \) and \( \text{COL}(17) = B \). By looking at Figure 1 we see that \( \text{COL}(26) \notin \{R, B\} \), hence \( \text{COL}(26) = G \). Again by looking at Figure 1 we have that \( \text{COL}(42) \notin \{B, G\} \), hence \( \text{COL}(42) = R \).

Note that we have shown that \( \text{COL}(1) = \text{COL}(42) \). More generally we have shown that, for all \( x \), \( \text{COL}(x) = \text{COL}(x + 41) \). Hence

\[ \text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41) = \text{COL}(1+41^2). \]
This contradicts COL being an \((x^2;3)\)-proper coloring.

The following theorem was proven by Matthew Jordan and William Gasarch.

**Theorem 5.2.**

1. \(W(x^2;3) = 29\).
2. For all \(a \in \mathbb{Z}\), \(W(ax^2;3) = 28a + 1\). This follows from Part 1 and Lemma 2.4

**Proof:**

\(W(x^2;3) \leq 29\): Assume, by way of contradiction, that there exists COL, a proper \((x^2,3)\)-coloring of \(\{1, \ldots, 29\}\). Figure 2 shows some constraints on COL: COL restricted to \(\{1, 10, 17, 26\}\) has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 2, \(\text{COL}(10) = \text{COL}(17)\). By similar reasoning one can show that
\[(\forall x)\{10 \leq x \leq 13 \rightarrow \text{COL}(x) = \text{COL}(x + 7)\}.

We refer to this fact as FORCE-SEVEN since the value of \(\text{COL}(x)\) forces the value of \(\text{COL}(x + 7)\).

![Figure 2: In any proper \((x^2,3)\)-coloring, \(\text{COL}(10) = \text{COL}(17)\)]
We can assume \( \text{COL}(10) = R \). Since \( 11 - 10 = 1^2 \) we know that \( \text{COL}(10) \neq \text{COL}(11) \), so we can assume \( \text{COL}(11) = B \).

17: By FORCE-SEVEN \( \text{COL}(17) = \text{COL}(10) = R \)

18: By FORCE-SEVEN \( \text{COL}(18) = \text{COL}(11) = B \).

\[
\begin{array}{cccccccccccc}
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
R & B & & & & & & R & B & & \\
\end{array}
\]

19: Since \( \text{COL}(10) = R \), \( \text{COL}(19) = \text{COL}(10 + 3^2) \neq R \). Since \( \text{COL}(18) = B \), \( \text{COL}(19) = \text{COL}(18 + 1^2) \neq B \). Hence \( \text{COL}(19) \notin \{ R, B \} \), so \( \text{COL}(19) = G \).

12: By FORCE-SEVEN \( \text{COL}(12) = \text{COL}(19) = G \).

\[
\begin{array}{cccccccccccc}
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
R & B & G & & & & & R & B & G & \\
\end{array}
\]

20: Since \( \text{COL}(11) = B \) and \( \text{COL}(19) = G \), \( \text{COL}(20) = R \).

13: By FORCE-SEVEN \( \text{COL}(13) = \text{COL}(20) = R \).

\[
\begin{array}{cccccccccccc}
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
R & B & G & R & & & R & B & G & R & \\
\end{array}
\]

Now we have that \( \text{COL}(17) = \text{COL}(13) = R \). But \( 17 - 13 = 2^2 \). This is a contradiction.

\( W(x^2, 3) \geq 29: \)

We present a proper 3-coloring:

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
B & G & R & G & R & B & G & B & G & R & B & G & B & G \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
R & B & R & B & G & R & B & R & B & G & R & G & R & B \\
\end{array}
\]
1. Even though we proved $W(x^2; 3) \leq 29$ it is of interest to see why the proper 3-coloring of [28] at the end of the proof of Theorem 5.2 cannot be extended to a 3-coloring of [29].

- 4 is colored $G$ and $29 - 4 = 25 = 5^2$. Hence 29 cannot be colored $G$.
- 25 is colored $R$ and $29 - 25 = 4 = 2^2$. Hence 29 cannot be colored $R$.
- 28 is colored $B$ and $29 - 28 = 1 = 1^2$. Hence 29 cannot be colored $B$.
- From the last three points, 29 cannot be colored $G$ or $R$ or $B$.

2. By Figure 2 we easily show $W(x^2; 3) \leq 68$: For $10 \leq x \leq 68$ then $\text{COL}(x) = \text{COL}(x + 7)$, so

$$\text{COL}(10) = \text{COL}(17) = \cdots = \text{COL}(59),$$

and note that $59 - 10 = 49 = 7^2$. This result is not as strong as $W(x^2; 3) \leq 29$; however, it has a less detailed proof.

6. Upper Bounds on $W(ax^2 + bx; 3)$

Definition 6.1.

(a) A coloring of $[w]$ has repeat distance $r$ if $x$ and $x + r$ have the same color, for all $1 \leq x \leq w - r$.

(b) A coloring of $[w]$ has repeat distance $r$ under one-sided boundary condition $b$ if $x$ and $x + r$ have the same color, for all $1 \leq x \leq w - r - b$.

(c) A coloring of $[w]$ has repeat distance $r$ under two-sided boundary condition $b$ if $x$ and $x + r$ have the same color, for all $b + 1 \leq x \leq w - r - b$.

Lemma 6.2. In any 3-coloring of $[w]$ with forbidden distances $s, t, s + t$, where $0 < s < t$:

(a) $2s + t$ is a repeat distance.

(b) $t - s$ is a repeat distance under two-sided boundary condition $s$. 

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(c) 3s is a repeat distance under one-sided boundary condition t.

Proof: Let \( u = s + t \).

(a) Consider a 3-coloring satisfying the conditions of the lemma. Let \( 1 \leq x \leq w - (2s + t) \). Without loss of generality, we can assume that \( x \) is R. Then \( x + s \) is not R, say B, and \( x + u = (x + s) + t \) cannot be R or B so it must be G. Then \((x + s) + u = (x + u) + s\) cannot be B or G so it must be R. Since \( x \) and \( x + u + s \) are both R, \((x + u + s) - x = u + s = 2s + t\) is a repeat distance.

(b) Consider a 3-coloring satisfying the conditions of the lemma. Let \( s < x \leq w - (t - s) - s \). Without loss of generality, we can assume that \( x \) is R. Then \( x - s \) is not R, say B, and \((x - s) + u = x + t\) cannot be R or B so it must be G. Then \((x - s) + t = (x + t) - s\) cannot be B or G, so it must be R. This process requires that \( x - s > 0 \) and \( x + t \leq n \). So \((x + t - s) - x = t - s\) is a repeat distance under two-sided boundary condition s.

(c) Take \( 2s + t \) from part (a) and subtract \( t - s \) from part (b). The repeat distance is \((2s + t) - (t - s) = 3s\). There is a one-sided boundary of size \((t - s) + s = t\) from one side of part (b).

Lemma 6.3. Assume \([w]\) has a proper 3-coloring \( \text{COL} \) where \( s \) is a forbidden distance and \( r \) is repeat distance under either one-sided or two-sided boundary condition b. If \( r|s \) then

\[ w \leq s + 2b + 1. \]

Proof: We prove the lemma for 2-sided boundary condition. The case of 1-sided boundary condition follows immediately.

Assume, by way of contradiction, that \( w \geq s + 2b + 2 \). By (a) the definition of repeat distance under two sided boundary condition, (b) \( r|s \), and (c) \( b + 1 + (\frac{r}{r} - 1)r \leq w - r - b \) (this is equivalent to \( w \geq 2b + s + 1 \) which follows from \( w \geq 2b + s + 2 \)) we have:
COL(b + 1) = COL(b + 1 + r) = COL(b + 1 + 2r) = \cdots

= COL(b + 1 + \left(\frac{s}{r} - 1\right) r = COL(b + 1 + \left(\frac{s}{r}\right) r) = COL(b + 1 + s).

But s is a forbidden distance so b + 1 and s + b + 1 cannot have the same color. Contradiction.

We use Lemma 6.3 to get upper bounds on several quadratic van der Waerden numbers. For one of them we have an exact value.

**Theorem 6.4.**

1. For a, b > 0 and a|b, \( W(ax^2 + bx; 3) \leq \frac{72b^2}{a} + 1. \)
2. \( W(x^2 + x; 3) = 73. \)

**Proof:**

1) Let \( p(x) = ax^2 + bx. \) Let \( x = \frac{5b}{a}, \quad y = \frac{6b}{a}, \quad z = \frac{8b}{a}. \)

Then
\[
p(x) = \frac{30b^2}{a}, \quad p(y) = \frac{42b^2}{a}, \quad p(z) = \frac{72b^2}{a}.
\]

Since \( p(x) + p(y) = p(z), \) by Lemma 6.2b, \( p(y) - p(x) = \frac{12b^2}{a} \) is a repeat distance under two-sided boundary condition \( \frac{30b^2}{a}. \) But \( p(\frac{3b}{a}) = \frac{12b^2}{a} \) is a forbidden distance. Thus, by Lemma 6.3, \( W(ax^2 + bx; 3) \leq \frac{12b^2}{a} + 2 \cdot \frac{30b^2}{a} + 1 = \frac{72b^2}{a} + 1. \)

2) By Part 1 \( W(x^2 + x; 3) \leq 73. \) We show \( W(x^2 + x; 3) \geq 73 \) by giving a \( (x^2 + x; 3) \)-proper coloring of \( \{1, \ldots, 72\}. \)

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We need one more lemma before getting an upper bounds for $W(p(x); 3)$ where $p(x) = ax^2 + bx$.

**Lemma 6.5.** Let $q(a, b) = (4a^3 + 4a^2 + 3a + 1)b^2$ and $r(a, b) = (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2$.

Then, for all $a, b \in \mathbb{Z}$, $\gcd(2q(a, b) + r(a, b), 3q(a, b)) \leq 18b^2$.

**Proof:** Let $a, b \in \mathbb{Z}$ and let $d = \gcd(q(a, b), r(a, b))$.

The reader can verify the following equation:

$$(−20a^4−12a^3−4a^2−10a−1)(4a^3+4a^2+3a+1)b^2+(20a^2−8a+7)(4a^5+8a^4+8a^3+6a^2+3a+1)b^2 = 6b^2.$$

Since $d$ divides the LHS, $d$ divides the RHS, hence

$$(\forall a, b \in \mathbb{Z})[\gcd((4a^3 + 4a^2 + 3a + 1)b^2, (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2 \leq 6b^2]$$

so

$$(\forall a, b \in \mathbb{Z})[\gcd(q(a, b), r(a, b)) \leq 6b^2].$$

Therefore:

$$\gcd(2q(a, b) + r(a, b), 3q(a, b)) \leq \gcd(3(2q(a, b) + r(a, b)) - 2(3q(a, b)), 3q(a, b))$$

$$= \gcd(3r(a, b), 3q(a, b)) = 3 \gcd(r(a, b), q(a, b)) \leq 18b^2.$$
Theorem 6.6. Let \( p(x) = ax^2 + bx \). Then \( W(p(x); 3) \leq O(|a^5b^2|) \).

Proof: We prove the theorem for \( a, b \geq 0 \). The other cases are similar.

Let

\[
x_0 = (2a + 1)b, \quad y_0 = (2a^2 + 2a + 1)b, \quad z_0 = (2a^2 + 2a + 2)b.
\]

Then

\[
\begin{align*}
p(x_0) &= (4a^3 + 4a^2 + 3a + 1)b^2 \\
p(y_0) &= (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2 \\
p(z_0) &= (4a^5 + 8a^4 + 12a^3 + 10a^2 + 6a + 2)b^2
\end{align*}
\]

Thus \( p(x_0) + p(y_0) = p(z_0) \). By Lemma 6.2b, \( 2p(x_0) + p(y_0) \) is a repeat distance, and by Lemma 6.2c, \( 3p(x_0) \) is a repeat distance under one-sided boundary condition \( p(y_0) \). By Lemma 6.5 we obtain that \( \gcd(2p(x_0) + p(y_0), 3p(x_0)) = db^2 \) for some constant \( d \leq 18 \). Thus there exists a linear combination over \( \mathbb{Z} \) of \( 2p(x_0) + p(y_0) \) and \( 3p(x_0) \) that sums to \( db^2 \). Since both quantities are \( > db^2 \) the linear combination has to have one positive coefficient and one negative coefficient. We assume that the coefficient of \( 2p(x_0) + p(y_0) \) is positive and the coefficient of \( 3p(x_0) \) is negative (the other case is similar). Hence there exists \( j, k \in \mathbb{N} \) such that \( j(2p(x_0) + p(y_0)) - k(3p(x_0)) = db^2 \). By starting at 1 and adding repeat distance \( 2p(x_0) + p(y_0) \) \( j \) times and subtracting repeat distance \( 3p(x_0) \) \( k \) times, we see that \( db^2 \) is also a repeat distance. Furthermore, by interspersing the adds and subtracts so that we subtract whenever the sum is greater than \( 2p(x_0) + p(y_0) \), the one-sided boundary condition is \( (2p(x_0) + p(y_0)) + p(y_0) = 2(p(x_0) + p(y_0)) \). Thus for any integer \( \alpha \), \( \alpha db^2 \) is a repeat distance with the one-sided boundary condition \( 2(p(x_0) + p(y_0)) \). But \( p(db) = ad^2b^2 + b^2d = (ad + 1)db^2 \) is a forbidden distance. So, \( W(p(x); 3) \leq p(db) + 2(p(x_0) + p(y_0)) = O(a^5b^2) \).

In Appendix B is a table of some exact values of \( W(ax^2 + bx; 3) \).
7. Upper Bounds on $W(x^2; 4)$

Recall that Figure 1 was the key to showing $W(x^2; 3) \leq 1682$. We now derive parameters for a new figure that will be the key to an upper bound on $W(x^2; 4)$.

We need to find $a, b, c, d, e, f, x, y, z \in \mathbb{N}^+$ such that the following figure can be drawn:

$$
\begin{align*}
\text{Figure 3: In any } (x^2; 4)\text{-proper coloring, } \text{COL}(1) &= \text{COL}(1 + w) \\
&
\end{align*}
$$

Hence we need to find solutions in $\mathbb{N}^+$ to the following system of equations:

\[
\begin{align*}
  x^2 + a^2 &= y^2 \\
  x^2 + b^2 &= z^2 \\
  y^2 + c^2 &= z^2 \\
  x^2 + d^2 &= w \\
  y^2 + e^2 &= w \\
  z^2 + f^2 &= w
\end{align*}
\]

The first three equations are overlapping Pythagorean triples—we have three numbers $(x, y, z)$ whose squares have all square pairwise differences. From the
first three equations one can derive the following:

\begin{align*}
  c^2 + f^2 &= e^2 \\
  b^2 + f^2 &= d^2 \\
  a^2 + c^2 &= b^2
\end{align*}

We give one example by deriving \( c^2 + f^2 = e^2 \): From \( y^2 + c^2 = z^2 \) and \( z^2 + f^2 = w \) we get \( y^2 + c^2 = w - f^2 \), and hence

\[ c^2 + f^2 = w - y^2. \]

From \( y^2 + e^2 = w \) we get that the left hand side is \( e^2 \). Hence

\[ c^2 + f^2 = e^2. \]

Since the first three equations are Pythagorean triples, they can be generated by using Euclid’s formula: all Pythagorean triples are of the form \((k(m^2 - n^2), k(2mn), k(m^2 + n^2))\) where \( \gcd(m, n) = 1 \) and \( m \not\equiv n \pmod{2} \). We can use the Farey sequence as an efficient algorithm to generate coprime pairs \( m, n \).

We used a computer program and obtained the following:

**Theorem 7.1.** \( W(x^2; 4) \leq 1 + (290,085,289)^2 = 84,149,474,894,213,522 \)

**Proof:**

Assume, by way of contradiction, that there exists \( \text{COL} \), a proper \((x^2; 4)\)-coloring of \([1 + (290,085,289)^2] \). Figure 4 shows some constraints on \( \text{COL} \): \( \text{COL} \) restricted to the numbers on the vertices has to be a proper 4-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 4 we know that

\[ \text{COL}(1) = \text{COL}(1 + 290,085,289^2). \]

More generally we have shown that, for all \( x \),

\[ \text{COL}(x) = \text{COL}(x + 290,085,289^2). \]
Figure 4: In any \((x^2; 4)\)-proper coloring, \(\text{COL}(1) = \text{COL}(1 + 290,085,290)\)

Hence

\[
\text{COL}(1) = \text{COL}(1+290,085,290) = \text{COL}(1+2 \times 290,085,290) = \cdots = \text{COL}(1+(290,085,290)^2).
\]

This contradicts \(\text{COL}\) being an \((x^2; 4)\)-proper coloring.  

Theorem 7.1 gave an enormous upper bound on \(W(x^2; 4)\). The proof was found by a computer program; however, it is a HS proof and human-verifiable. Four colors seems to be at the limit of what computers can find. That is, we have been unable to use a program to find a human-verifiable proof for a bound on \(W(x^2; 5)\).
Usually a HS proof gives better bounds than a proof that uses advanced mathematics. However, our HS proof of a bound on $W(x^2; 4)$ gives such a large bound that its possible a proof using more advanced mathematics would yield a better result. In particular, its possible that if the proofs of the results of Sarkozy [8], Pintz-Steiger-Szemerédi [15], or Harnel-Lyall-Rice [17] were looked at more carefully then one could obtain better bounds for $W(x^2; c)$ for some small values of $c$. However, these would not be HS proofs.

8. Upper Bounds on $W(Ax^2 + Bx; 4)$

To find upper bounds on $W(Ax^2 + Bx; 4)$ we have several overlapping equations of the form

$$(Ax^2 + Bx) + (Ay^2 + By) = (Az^2 + Bz).$$

We need a way to generate such triples $(x, y, z)$ much like the generation of Pythagorean triples. First, we use the quadratic formula to express $z$ in terms of $x$ and $y$.

$$z = \frac{-B + \sqrt{4A^2(x^2 + y^2) + 4AB(x + y) + B^2}}{2A}$$

We rewrite as

$$4A^2(x^2 + y^2) + 4AB(x + y) + B^2 = (2Az + B)^2.$$

Simple algebra allows us to rewrite this as:

$$(2Ax + B)^2 + (2Ay + B)^2 = (2Az + B)^2 + B^2.$$

If $m = 2Ax + B$, $n = 2Ay + B$, and $k = (2Az + B)$ then we can rewrite this as $m^2 + n^2 = k^2 + B^2$. A parameterization of $m^2 + n^2 = k^2 + B^2$ would imply one for $(x, y, z)$, and luckily this equation is easier. Using the Bramagupta-Fibonacci identity with $bc - ad = B$, we get:

$$(ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + B^2$$

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So, with parameters $a, b, c, d$ and some tedious algebra we get

$$x = \frac{ac - bd - B}{2A}, \quad y = \frac{ad + bc - B}{2A}, \quad z = \frac{ac + bd - B}{2A}$$

with constraints $bc - ad = B, \ ac - bd > B, \ 2A|ac - bd - B, \ 2A|ad + bc - B$.

Rather than searching all $(a, b, c, d)$, we can eliminate parts of the parameter space that do not contain solutions. For fixed $a$ and $d$, the first constraint implies that $bc$ is some factorization of $ad + B$. We can pre-compute a table of factorizations and use that to cut the search space down to almost $O(n^2)$. You can see the code for this on GitHub at [https://github.com/zaprice/polyvdw](https://github.com/zaprice/polyvdw)

We can get bounds for $W(x^2 + Bx; 4)$ with this method with rather large values of $B$, but only a few bounds for the more general $Ax^2 + Bx$ case; if such configurations exist, it seems the numbers involved are much larger. See Appendix C for some of the upper bounds we have. We note two things about these upper bounds:

1. The largest upper bound on $W(x^2 + Bx; 4)$ that we found was when $B = 0$.
   Note that these are just the upper bounds we found. It is not clear how the real values compares.

2. For $W(2x^2 + Bx; 4)$ and $W(3x^2 + Bx; 4)$ the $B$ for which we could find an upper bound seem scattered and arbitrary. For example, we were not able to find an upper bound for any of $W(2x^2 + Bx; 4)$ for $0 \leq B \leq 56$, but were able to for 57. And then not again until $B = 95$. Again, this may be a limit to our methods and not a statement about the actual values of $W(2x^2 + Bx; 4)$.

Acknowledgement

We thank Sean Prediville and Alex Rice for discussions about prior known upper bounds on $W(p; c)$.

We thank Steven Brown, Nathan Hayes, Emily Kaplitz, Yuan Shen, and Zan Xu for proofreading and discussion.
References


Appendix A. Some Exact Values of $W(ax^2 + bx; 2)$

We present a table of $W(p(x); 2)$ for $p(x) = ax^2 + bx$ for $0 \leq a \leq 10$ and $-10 \leq b \leq 10$. 

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& \begin{tabular}{c}
$b$ \\
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& $W(ax^2 + bx; 2)$ \\
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The values for \(a, b \geq 0\) were obtained by using our formulas for an upper bound and then searching for a 2-coloring for the lower bound.

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<td>25</td>
<td>31</td>
<td>37</td>
<td>43</td>
<td>73</td>
<td>53</td>
<td>57</td>
<td>61</td>
<td>65</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>25</td>
<td>31</td>
<td>49</td>
<td>41</td>
<td>45</td>
<td>51</td>
<td>85</td>
<td>61</td>
<td>65</td>
<td>71</td>
</tr>
<tr>
<td>8</td>
<td>17</td>
<td>29</td>
<td>33</td>
<td>39</td>
<td>41</td>
<td>49</td>
<td>53</td>
<td>59</td>
<td>97</td>
<td>69</td>
<td>73</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
<td>37</td>
<td>37</td>
<td>37</td>
<td>47</td>
<td>73</td>
<td>55</td>
<td>61</td>
<td>67</td>
<td>109</td>
<td>77</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
<td>35</td>
<td>49</td>
<td>45</td>
<td>49</td>
<td>51</td>
<td>57</td>
<td>65</td>
<td>69</td>
<td>75</td>
<td>121</td>
</tr>
</tbody>
</table>

The numbers tend to increase with increasing \(a\) and \(|b|\). Some of the diagonals have patterns which likely can be used to make conjectures that are almost surely true. For example:

\[ (\forall a \geq 0)[W(ax^2 - (a - 1)x; 2) = 2a + 3]. \]
Appendix B. Some Exact Values of $W(ax^2 + bx; 3)$

We present a table of $W(p(x); 3)$ for $p(x) = ax^2 + bx$ for $0 \leq a \leq 5$ and $-5 \leq b \leq 5$.

The values were obtained by computer.

<table>
<thead>
<tr>
<th>$b$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>16</td>
<td>1</td>
<td>64</td>
<td>61</td>
<td>217</td>
<td>1</td>
</tr>
<tr>
<td>-4</td>
<td>13</td>
<td>1</td>
<td>1</td>
<td>91</td>
<td>1</td>
<td>289</td>
</tr>
<tr>
<td>-3</td>
<td>10</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>135</td>
<td>171</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>68</td>
<td>97</td>
<td>171</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>1</td>
<td>49</td>
<td>105</td>
<td>190</td>
<td>183</td>
</tr>
</tbody>
</table>

Appendix C. Some Upper Bounds on $W(ax^2 + bx; 4)$

We give bounds for $W(g; 4)$ where $g$ is of the form $Ax^2 + Bx$. Only bounds for coprime coefficients $(A, B)$ are presented. Each row of the table gives $g$, $x, y, z, w$ (as in Figure C.5), and the bound. We give four such tables.
Figure C.5: In any $(g(x); 4)$-proper coloring, $\text{COL}(1) = \text{COL}(1 + w)$
Table for $x^2 + Bx$ where $0 \leq B \leq 20$. 

<table>
<thead>
<tr>
<th>$g$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
<th>$W(g(x); 4) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>10,608</td>
<td>13,108</td>
<td>16,133</td>
<td>290,085,289</td>
<td>84,149,474,894,213,522</td>
</tr>
<tr>
<td>$x^2 + x$</td>
<td>299</td>
<td>302</td>
<td>327</td>
<td>113,262</td>
<td>12,828,393,907</td>
</tr>
<tr>
<td>$x^2 + 2x$</td>
<td>91</td>
<td>127</td>
<td>211</td>
<td>257,463</td>
<td>66,287,711,296</td>
</tr>
<tr>
<td>$x^2 + 3x$</td>
<td>35</td>
<td>43</td>
<td>53</td>
<td>3,308</td>
<td>10,952,789</td>
</tr>
<tr>
<td>$x^2 + 4x$</td>
<td>80</td>
<td>84</td>
<td>92</td>
<td>10,197</td>
<td>104,019,598</td>
</tr>
<tr>
<td>$x^2 + 5x$</td>
<td>70</td>
<td>81</td>
<td>100</td>
<td>11,250</td>
<td>126,618,751</td>
</tr>
<tr>
<td>$x^2 + 6x$</td>
<td>70</td>
<td>86</td>
<td>106</td>
<td>13,232</td>
<td>175,165,217</td>
</tr>
<tr>
<td>$x^2 + 7x$</td>
<td>638</td>
<td>785</td>
<td>923</td>
<td>988,338</td>
<td>976,818,920,611</td>
</tr>
<tr>
<td>$x^2 + 8x$</td>
<td>160</td>
<td>168</td>
<td>184</td>
<td>40,788</td>
<td>1,663,987,249</td>
</tr>
<tr>
<td>$x^2 + 9x$</td>
<td>35</td>
<td>37</td>
<td>44</td>
<td>3,242</td>
<td>10,539,743</td>
</tr>
<tr>
<td>$x^2 + 10x$</td>
<td>144</td>
<td>150</td>
<td>165</td>
<td>36,075</td>
<td>1,301,766,376</td>
</tr>
<tr>
<td>$x^2 + 11x$</td>
<td>364</td>
<td>472</td>
<td>727</td>
<td>1,263,252</td>
<td>1,595,819,511,277</td>
</tr>
<tr>
<td>$x^2 + 12x$</td>
<td>140</td>
<td>172</td>
<td>212</td>
<td>52,928</td>
<td>2,802,008,321</td>
</tr>
<tr>
<td>$x^2 + 13x$</td>
<td>119</td>
<td>129</td>
<td>143</td>
<td>38,016</td>
<td>1,445,710,465</td>
</tr>
<tr>
<td>$x^2 + 14x$</td>
<td>66</td>
<td>96</td>
<td>135</td>
<td>25,395</td>
<td>645,261,556</td>
</tr>
<tr>
<td>$x^2 + 15x$</td>
<td>120</td>
<td>138</td>
<td>215</td>
<td>54,364</td>
<td>2,956,259,957</td>
</tr>
<tr>
<td>$x^2 + 16x$</td>
<td>75</td>
<td>99</td>
<td>141</td>
<td>45,177</td>
<td>2,041,684,162</td>
</tr>
<tr>
<td>$x^2 + 17x$</td>
<td>123</td>
<td>165</td>
<td>255</td>
<td>232,908</td>
<td>54,250,095,901</td>
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<tr>
<td>$x^2 + 18x$</td>
<td>70</td>
<td>74</td>
<td>88</td>
<td>12,968</td>
<td>168,402,449</td>
</tr>
<tr>
<td>$x^2 + 19x$</td>
<td>65</td>
<td>66</td>
<td>69</td>
<td>6,852</td>
<td>47,080,093</td>
</tr>
<tr>
<td>$x^2 + 20x$</td>
<td>84</td>
<td>96</td>
<td>115</td>
<td>24,261</td>
<td>589,081,342</td>
</tr>
</tbody>
</table>
Table for $x^2 + Bx$ where $1980 \leq B \leq 2000$.  

<table>
<thead>
<tr>
<th>$g$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
<th>$W(g(x);4) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 1,980x$</td>
<td>1,683</td>
<td>2,145</td>
<td>2,915</td>
<td>25,524,829</td>
<td>651,567,434,640,662</td>
</tr>
<tr>
<td>$x^2 + 1,981x$</td>
<td>1,674</td>
<td>1,735</td>
<td>2,026</td>
<td>14,236,652</td>
<td>202,710,462,976,717</td>
</tr>
<tr>
<td>$x^2 + 1,982x$</td>
<td>1,248</td>
<td>1,495</td>
<td>1,731</td>
<td>6,882,723</td>
<td>47,385,517,451,716</td>
</tr>
<tr>
<td>$x^2 + 1,983x$</td>
<td>3,498</td>
<td>3,549</td>
<td>3,664</td>
<td>24,967,678</td>
<td>623,434,455,617,159</td>
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<tr>
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<td>860</td>
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<td>2,585</td>
<td>12,424,497</td>
<td>154,392,775,905,058</td>
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<tr>
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<td>867</td>
<td>1,098</td>
<td>2,365</td>
<td>11,200,200</td>
<td>125,466,712,437,001</td>
</tr>
<tr>
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<td>1,900</td>
<td>2,432</td>
<td>2,908</td>
<td>19,712,552</td>
<td>388,623,855,480,977</td>
</tr>
<tr>
<td>$x^2 + 1,987x$</td>
<td>3,048</td>
<td>3,393</td>
<td>3,987</td>
<td>39,165,018</td>
<td>1,533,976,455,831,091</td>
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<tr>
<td>$x^2 + 1,988x$</td>
<td>508</td>
<td>738</td>
<td>1,194</td>
<td>6,489,996</td>
<td>42,132,950,192,065</td>
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<tr>
<td>$x^2 + 1,989x$</td>
<td>2,023</td>
<td>2,288</td>
<td>3,094</td>
<td>18,950,528</td>
<td>359,160,204,078,977</td>
</tr>
<tr>
<td>$x^2 + 1,990x$</td>
<td>1,364</td>
<td>1,610</td>
<td>2,100</td>
<td>13,163,856</td>
<td>173,313,300,862,177</td>
</tr>
<tr>
<td>$x^2 + 1,991x$</td>
<td>1,330</td>
<td>1,519</td>
<td>1,814</td>
<td>7,817,030</td>
<td>61,121,521,727,631</td>
</tr>
<tr>
<td>$x^2 + 1,992x$</td>
<td>975</td>
<td>1,065</td>
<td>1,871</td>
<td>10,120,498</td>
<td>102,444,639,800,021</td>
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<tr>
<td>$x^2 + 1,993x$</td>
<td>1,985</td>
<td>2,349</td>
<td>4,373</td>
<td>68,596,488</td>
<td>4,705,614,878,734,729</td>
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<tr>
<td>$x^2 + 1,994x$</td>
<td>1,246</td>
<td>1,350</td>
<td>1,716</td>
<td>8,551,440</td>
<td>73,144,177,644,961</td>
</tr>
<tr>
<td>$x^2 + 1,995x$</td>
<td>891</td>
<td>1,185</td>
<td>1,464</td>
<td>10,543,450</td>
<td>111,185,372,085,251</td>
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<tr>
<td>$x^2 + 1,996x$</td>
<td>705</td>
<td>995</td>
<td>1,793</td>
<td>7,390,317</td>
<td>54,631,536,433,222</td>
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<tr>
<td>$x^2 + 1,997x$</td>
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<td>1,136</td>
<td>1,391</td>
<td>8,040,026</td>
<td>64,658,074,012,599</td>
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<tr>
<td>$x^2 + 1,998x$</td>
<td>1,292</td>
<td>1,732</td>
<td>3,704</td>
<td>39,649,768</td>
<td>1,572,183,322,690,289</td>
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<tr>
<td>$x^2 + 1,999x$</td>
<td>1,235</td>
<td>1,757</td>
<td>2,789</td>
<td>14,633,322</td>
<td>214,163,364,766,363</td>
</tr>
<tr>
<td>$x^2 + 2,000x$</td>
<td>184</td>
<td>280</td>
<td>984</td>
<td>5,592,000</td>
<td>31,281,648,000,001</td>
</tr>
</tbody>
</table>
### Table for $2x^2 + Bx$ for assorted $B$.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
<th>$W(g(x); 4) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x^2 + 57x$</td>
<td>3,969</td>
<td>4,035</td>
<td>4,295</td>
<td>38,199,155</td>
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</tr>
<tr>
<td>$2x^2 + 95x$</td>
<td>707</td>
<td>758</td>
<td>1,008</td>
<td>14,365,638</td>
<td>412,744,475,029,699</td>
</tr>
<tr>
<td>$2x^2 + 171x$</td>
<td>11,907</td>
<td>12,105</td>
<td>12,885</td>
<td>343,792,395</td>
<td>236,386,480,508,171,596</td>
</tr>
<tr>
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<td>2,121</td>
<td>2,274</td>
<td>3,024</td>
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<td>6,282</td>
<td>131,899,720</td>
<td>34,795,139,672,913,721</td>
</tr>
<tr>
<td>$2x^2 + 455x$</td>
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<td>3,663</td>
<td>4,170</td>
<td>39,229,128</td>
<td>3,077,866,816,534,009</td>
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<tr>
<td>$2x^2 + 511x$</td>
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<td>3,367</td>
<td>6,282</td>
<td>131,899,720</td>
<td>34,795,139,672,913,721</td>
</tr>
<tr>
<td>$2x^2 + 627x$</td>
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<td>44,385</td>
<td>47,245</td>
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<td>5,834,090,064,188,269,204</td>
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<tr>
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<td>1,303</td>
<td>2,920</td>
<td>87,446,025</td>
<td>15,293,684,970,651,376</td>
</tr>
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<td>5,548</td>
<td>7,087</td>
<td>13,262</td>
<td>530,042,423</td>
<td>561,890,393,545,693,524</td>
</tr>
<tr>
<td>$2x^2 + 1,011x$</td>
<td>5,164</td>
<td>6,568</td>
<td>9,889</td>
<td>318,517,859</td>
<td>202,907,575,025,443,212</td>
</tr>
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<td>$2x^2 + 1,153x$</td>
<td>12,705</td>
<td>12,726</td>
<td>12,970</td>
<td>352,488,525</td>
<td>248,496,726,932,620,576</td>
</tr>
<tr>
<td>$2x^2 + 1,199x$</td>
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<td>8,710</td>
<td>9,748</td>
<td>221,108,295</td>
<td>97,778,017,806,722,272</td>
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<tr>
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<td>14,355</td>
<td>22,244</td>
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<td>2,703,804,197,637,349,126</td>
</tr>
<tr>
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<td>26,105</td>
<td>28,172</td>
<td>1,638,880,116</td>
<td>5,371,858,201,423,377,829</td>
</tr>
<tr>
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<td>10,989</td>
<td>12,510</td>
<td>353,062,156</td>
<td>249,306,248,279,579,689</td>
</tr>
<tr>
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<td>1,174</td>
<td>1,379</td>
<td>58,465,486</td>
<td>6,836,511,407,576,467</td>
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<tr>
<td>$2x^2 + 1,545x$</td>
<td>11,298</td>
<td>11,815</td>
<td>12,860</td>
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</tr>
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<td>$2x^2 + 1,685x$</td>
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<td>10,968</td>
<td>11,570</td>
<td>289,144,125</td>
<td>167,209,137,251,881,876</td>
</tr>
<tr>
<td>$2x^2 + 1,753x$</td>
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<td>5,236</td>
<td>8,232</td>
<td>181,967,394</td>
<td>66,224,583,947,144,155</td>
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<td>$2x^2 + 1,851x$</td>
<td>50,031</td>
<td>51,441</td>
<td>55,164</td>
<td>6,379,649,159</td>
<td>7,612,882,297,751,201,408</td>
</tr>
<tr>
<td>$2x^2 + 1,913x$</td>
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<td>3,366</td>
<td>5,324</td>
<td>81,424,299</td>
<td>13,259,988,699,966,790</td>
</tr>
</tbody>
</table>
Table for $3x^2 + Bx$ for assorted $B$.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
<th>$W(g(x);4) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + x$</td>
<td>42.273</td>
<td>42.660</td>
<td>43.375</td>
<td>5,738,872,934</td>
<td>6,570,267,294,984,419,923</td>
</tr>
<tr>
<td>$3x^2 + 143x$</td>
<td>13.244</td>
<td>13.332</td>
<td>13.442</td>
<td>554,651,696</td>
<td>922,915,590,942,221,777</td>
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<tr>
<td>$3x^2 + 172x$</td>
<td>4.452</td>
<td>4.712</td>
<td>5.189</td>
<td>88,862,311</td>
<td>23,689,546,233,099,656</td>
</tr>
<tr>
<td>$3x^2 + 200x$</td>
<td>1.896</td>
<td>2.204</td>
<td>5.004</td>
<td>115,177,723</td>
<td>39,797,746,661,938,788</td>
</tr>
<tr>
<td>$3x^2 + 235x$</td>
<td>11.155</td>
<td>11.270</td>
<td>11.610</td>
<td>583,594,418</td>
<td>1,021,747,471,306,964,403</td>
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<tr>
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<td>9.322</td>
<td>11.610</td>
<td>16.903</td>
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<td>3,797,003,080,080,107,670</td>
</tr>
<tr>
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<td>9.424</td>
<td>10.378</td>
<td>355,449,244</td>
<td>379,032,617,455,054,545</td>
</tr>
<tr>
<td>$3x^2 + 361x$</td>
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<td>4.658</td>
<td>7.703</td>
<td>397,333,094</td>
<td>473,620,906,200,085,443</td>
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<td>3.792</td>
<td>4.408</td>
<td>10.008</td>
<td>460,710,892</td>
<td>636,763,762,306,663,793</td>
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<td>$3x^2 + 407x$</td>
<td>2.806</td>
<td>3.401</td>
<td>6.131</td>
<td>122,898,626</td>
<td>45,312,266,837,804,411</td>
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<td>2.077</td>
<td>2.829</td>
<td>5.839</td>
<td>392,773,686</td>
<td>462,813,667,064,838,421</td>
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<td>7.616</td>
<td>9.244</td>
<td>12.716</td>
<td>515,261,395</td>
<td>796,483,183,467,963,476</td>
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<td>9.400</td>
<td>9.408</td>
<td>9.451</td>
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<td>224,693,838,986,259,448</td>
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<td>$3x^2 + 592x$</td>
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<td>16.472</td>
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<td>52.351</td>
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<td>18.848</td>
<td>20.756</td>
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<td>3.278</td>
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<td>2.847</td>
<td>3.082</td>
<td>3.524</td>
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<td>66,520,245,642,541,737</td>
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<td>6.802</td>
<td>12.262</td>
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<td>724,995,869,246,944,305</td>
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<td>2.799</td>
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<td>18.902</td>
<td>1,094,699,196</td>
<td>3,595,100,206,474,645,201</td>
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