High School Proofs for Better Bounds on the Quadratic Van der Waerden Numbers

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Abstract

A corollary of the polynomial Van de Waerden theorem is that, for any polynomial $p(x) \in \mathbb{Z}[x]$ with constant term 0, for any $c \in \mathbb{N}$, there exists W such that, for all c-colorings of $\{1, \ldots, W\}$ there exists a, d such that a and $a+p(d)$ are the same color. The proof of the polynomial Van de Waerden theorem, and even of these corollaries, is difficult and gives enormous upper bounds for W. We consider just quadratic polynomials. For $c = 2, 3$ we obtain reasonable bounds, and for $c = 4$ for some quadratics we obtain reasonable bounds.

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1. Introduction

We use the following standard notation and definitions.

Def 1.1. Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{N}^+$ be the set of positive integers. Let $[W]$ be the set $\{1, \ldots, W\}$ (where $W \in \mathbb{N}$).

We also use the following informal definition.

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Def 1.2. A *High School Proof (HS Proof)* is a proof that can be explained to a bright high school student. In this paper all such proofs will be purely combinatorial as well.

We use the term *High School Proof* since (1) the terms *elementary* is ambiguous, and (2) the term *Combinatorial* is not quite right since (a) the rather difficult proof of Szemeredi's Theorem is combinatorial, and (b) the rather difficult proof of Gower's bound is mostly combinatorial.

Recall van der Waerden's Theorem [1, 2] (see also the books by Graham-Rothchild-Spencer [3] and Landman-Robertson [4]).

Theorem 1.3. *For any $k \in \mathbb{N}$, for any $c \in \mathbb{N}$, there exists $W = W(k, c)$, such that for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a + d, \ldots, a + (k - 1)d$ are all the same color.*

The original proof by van der Waerden was HS but yielded bounds on W that were not primitive recursive [3]. Shelah [5] gave a HS proof that yielded primitive recursive bounds on W. These bounds were still quite large in that they really cannot be written down nicely. Gowers [6] gave a non-HS proof that that yielded bounds that can be written down:

$$W(k, c) \leq 2^{2^{c^{2^{2^{k+9}}}}}$$

We discuss a known generalization of van der Waerden's theorem. Note that the conclusion of van der Waerden's theorem is that

$$a, a + d, a + 2d, \ldots, a + (k - 1)d \text{ are the same color.}$$

Can we replace $d, 2d, \ldots, (k - 1)d$ by other functions of d? Yes. We can replace them with polynomials in $\mathbb{Z}[x]$ that have no constant term. Here is the Polynomial van der Waerden Theorem:

Theorem 1.4. *Let $p_1(x), \ldots, p_k(x) \in \mathbb{Z}[x]$ such that, for $1 \leq i \leq k$, $p_i(0) = 0$. Let $c \in \mathbb{N}$. Then there exists $W = W(p_1(x), \ldots, p_k(x); c)$ such that, for any c-*

coloring of $[W]$*, there exists* $a, d \in \mathbb{N}$*,* $d \neq 0$*, such that* $a, a+p_1(d), \ldots, a+p_k(d)$ *are all the same color.*

For $k = 1$, this theorem was proven independently by Furstenberg [7] and Sárközy [8]. Bergelson and Leibman [9] proved the general result using ergodic methods (not a HS proof). These proofs yielded no upper bounds on $W(p_1(x), \ldots, p_k(x); c)$. Walters [10] obtained a HS proof of Theorem 1.4, but the bounds on W were not primitive recursive. Shelah [11] gave a (non HS) proof that yielded primitive recursive bounds on W. These bounds were still quite large in that they really cannot be written down nicely. Nobody has obtained a proof that yields bounds one can write down.

Peluse [12] has the best known upper bounds for sets of polynomials of distinct degrees. Peluse and Prediville [13] have the best known upper bounds for $W(x^2, x^2 + x; c)$. These proofs are not HS. With some effort one can write down these bounds in many cases (similar to Gowers bound on van der Warden Numbers).

We are interested in the case of $W(ax^2 + bx; c)$ where $c = 2, 3, 4$. Furstenberg's proof showed that $W(x^2; c)$ exists; however, his proof gave no upper bounds. Sárközy's proof showed that $W(x^2; c) \leq 2^{O(c^3)}$. Pintz, Steiger, and Szemeredi [14] (see also [15] for exposition) showed that $W(x^2; c) \leq 2^{O(c^{0.0001})}$. The 0.0001 can be replaced with any smaller constant; however, in that case the constant associated with the big-O will increase. It is possible that either Sárközy's proof of $W(x^2; c) \leq 2^{O(c^3)}$ or Pintz, Steiger, and Szemeredi proof of $W(x^2; c) \leq 2^{O(c^{0.0001})}$ could be modified with a fixed value of c such as 4. That may lead to an improvement on our bound on $W(x^2; 4)$; however, such a proof would not be HS.

Harnel, Lyall, and Rice [16] showed that there exists a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ such that

$$W(ax^2 + bx; c) \leq 2^{f(a,b)c^{0.0001}}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function f will be bigger).

Later Rice [15] showed that, for all k, there exists a function $f : \mathbb{Z}^k \rightarrow \mathbb{N}$ such that

$$W(a_kx^k + \cdots + a_1x; c) \leq 2^{f(a_k,\ldots,a_1)c^{0.0001}}$$

(the 0.0001 can be replaced with any smaller constant; however, in that case the function f will be bigger). Rice [17] later obtained the following more precise result: for all $\epsilon > 0$, for all $a_1, \ldots, a_k \in \mathbb{Z}$, for $J = |a_1| + \cdots + |a_k|$:

$$W(a_kx^k + \cdots + a_1x; c) \leq 2^{2^{2^{2^{100k^2/\epsilon}}}} + 2^{2^{2^{(100k^4 \log J)^{100}}}} + 2^{c^\epsilon}$$

In summary, the known bounds on $W(ax^2 + bx; c)$ are large.

In this paper we show that, for some $p(x) \in \mathbb{Z}[x]$ and $c = 2, 3, 4$, one can obtain sane bounds on $W(p(x); c)$. Our proofs will be purely combinatorial and much easier than those of Walters, Shelah, and Peluse. We hasten to point out that they proved the full poly van der Warden theorem whereas we only prove it in very special cases.

We will show the following.

- For all $a \in \mathbb{Z}$, $W(ax; c) = |ac| + 1$.
- For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \leq 12|a| + 6|b|$. We actually obtain more precise bounds than that depending on how a, b are related to each other. In Appendix A is a table of some exact values of $W(ax^2 + bx; 2)$.
- For all $a \in \mathbb{N}$, $a \geq 1$, $W(ax^2 + (a-1)x^2) = 8a - 3$.
- $W(x^2; 3) = 29$ and, for all $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$.
- For $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 3) \leq O(a^2b^5)$. In Appendix B is a table of some exact values of $W(ax^2 + bx; 3)$.
- $W(x^2; 4) \leq$ 84,149,474,894,213,522. In Appendix C is a table of some upper bounds on $W(ax^2 + bx; 4)$.

2. Preliminaries

Def 2.1. Let $c \in \mathbb{N}^+$ and $W \in \mathbb{N}^+$.

1. A *c-coloring of* $[W]$ is a mapping $[W] \to [c]$.
2. Let $p(x) \in \mathbb{Z}[x]$. A $(p(x);c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$, if $y - x = p(d)$ for some $d \in N^+$, then x and y have different colors. When the context is clear, we will often write *proper c-coloring* or simply *proper coloring*.

Note that the polynomial van der Waerden number, $W = W(p(x);c)$, is the least number such that there is no $(p(x);c)$-proper coloring of $[W]$.

Although we care about proper $(p(x);c)$-colorings, we need a more general notion:

Def 2.2. Let $F \subseteq \mathbb{Z}$, $c \in \mathbb{N}^+$, and $W \in \mathbb{N}^+$.

- An $(F;c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$ with $y - x \in F$, x and y have different colors.
- $W = W(F;c)$ is the least number such that there is no $(F;c)$-proper coloring of $[W]$. If no such number exists, we set $W(F;c) = \infty$.
- In the above definitions F is the set of *forbidden distances*. We will use this term for polynomial VDW numbers as well. For example, if looking at $W(3x^2;c)$ the forbidden distances are $3 \times 1^2, 3 \times 2^2, \ldots$.

We leave the following easy lemma to the reader.

Lemma 2.3. *Let* $c \in \mathbb{N}^+$.

1. *If* $0 \in F$ *then* $W(F;c) = 1$.
2. *Assume* $f \in F$. *Let* $F' = F \cup \{-f\}$. *Then* $W(F;c) = W(F';c)$.

Lemma 2.4. *Let* $p(x) \in \mathbb{Z}[x]$, $a \in \mathbb{Z}$, *and* $c \in \mathbb{N}$. *Then* $W(ap(x);c) = a(W(p(x);c) - 1) + 1$.

Proof:

1) $W(ap(x);c) \leq a(W(p(x);c)-1)+1$:

Assume, by way of contradiction, that $W(ap(x);c) \geq a(W(p(x);c)-1)+2$. Hence there exists COL, an $(ap(x);c)$-proper coloring of $[a(W(p(x);c)-1)+1]$. Note that, for all x, $ap(x)$ is a forbidden distance for COL.

We use COL to define COL$'$, a proper $(p(x);c)$-coloring of $[W(p(x);c)]$; which contradicts the definition of $W(p(x);c)$.

For $1 \leq i \leq W(p(x);c)$ let

$$\mathrm{COL}'(i) = \mathrm{COL}(a(i-1)+1).$$

Note that

$$\mathrm{COL}'(j) = \mathrm{COL}'(i) \text{ iff } \mathrm{COL}(a(j-1)+1) = \mathrm{COL}(a(i-1)+1).$$

Suppose $j-i$ is a forbidden distance for COL$'$. Then there exists x such that $j-i=p(x)$. Then

$$a(j-1)+1-a(i-1)+1 = a(j-i) = ap(x), \text{a forbidden distance for COL.}$$

Hence $\mathrm{COL}(a(j-1)+1) \neq \mathrm{COL}(a(i-1)+1)$, so $\mathrm{COL}'(j) \neq \mathrm{COL}'(i)$. Therefore COL$'$ is a proper $(p(x);c)$-coloring of $[W(p(x);c)]$.

2) $W(ap(x);c) \geq a(W(p(x);c)-1)+1$:

Let COL$'$ be a proper $(p(x);c)$-coloring of $[X]$. The reader can easily verify that COL, defined below, is a proper $(ap(x);c)$-coloring of $[aX]$.

- Color $1,\ldots,a$ with COL$'(1)$.
- Color $a+1,\ldots,2a$ with COL$'(2)$.
- $\vdots$
- Color $(X-1)a+1,\ldots,Xa$ with COL$'(X)$.

Take $X = W(p(x);c)$.

∎

3. Upper bounds on $W(ax;2)$

For completeness we cover linear polynomials, for which we obtain a complete solution.

Theorem 3.1. *Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}^+$. Then*

$$W(ax;c) = |ac| + 1 \ .$$

Proof: By Lemma 2.3 (1) we have the case of $a = 0$, and (2) we can assume that $|a|$ is a forbidden distance.

$W(ax;c) \le |ac| + 1$:

By setting $x = 1, 2, \ldots, c$ we get forbidden distances $|a|, |2a|, \ldots, |ca|$. So $1, |a|+1, |2a|+1, \ldots, |ca|+1$ must all be different colors, but there are only c colors.

$W(ax;c) \ge |ac| + 1$:

We can properly c-color $[ca]$:

- Color $1, \ldots, |a|$ with 1.
- Color $|a|+1, \ldots, |2a|$ with 2.
- $\vdots$
- Color $|(c-1)a + 1, \ldots, |ca|$ with c.

∎

4. Upper Bounds on $W(ax^2 + bx;2)$

Theorem 4.1. *Let $a, b \in \mathbb{N}$ with $a \ge 1$ and $b \ge 0$.*

1. $W(ax^2 + bx, 2) \le 12a + 6b + 1$.
2. *If $b \ge 3a$ then $W(-ax^2 + bx, 2) \le 6b - 12a + 1$.*
3. *If $2a \le b \le 3a$ then $W(-ax^2 + bx, 2) \le 3b - 3a + 1$.*
4. *If $a \le b \le 2a$ then $W(-ax^2 + bx, 2) \le 9a - 3b + 1$.*

5. *If* $0 \leq b \leq a$ *then* $W(-ax^2 + bx, 2) \leq 12a - 6b + 1$.
6. *One can obtain bounds for* $W(ax^2 - bx, 2)$ *easily since it equals* $W(-ax^2 + bx, 2)$.

Proof: Let d be a forbidden distance. For all y such that, $y, y+d, y+2d, y+3d$ are colored

$$\mathrm{COL}(y) = R \rightarrow \mathrm{COL}(y+d) = B \rightarrow \mathrm{COL}(y+2d) = R \rightarrow \mathrm{COL}(y+3d) = B.$$

Hence $3d$ is also a forbidden distance.

1) $W(ax^2 + bx; 2)$. By plugging in $x = 1, 2, 3$ we find forbidden distances:

$$\{a+b, 4a+2b, 9a+3b\}.$$

We will use the following forbidden distances:

$$\{3a+3b, 12a+6b, 9a+3b\}.$$

Assume, by way of contradiction that there is a proper $W(x^2; 2)$-coloring of $[12a + 6b + 1]$. We can assume that $\mathrm{COL}(1) = R$. Note that

$$\mathrm{COL}(1) = R \rightarrow \mathrm{COL}(1 + (3a+3b)) = B \rightarrow \mathrm{COL}(1 + (3a+3b) + (9a+3b)) = R.$$

We simplify to obtain $\mathrm{COL}(12a + 6b + 1) = R$.

$$\mathrm{COL}(12a + 6b + 1) = R \rightarrow \mathrm{COL}(12a + 6b + 1 - (12a + 6b)) = B.$$

We simplify to obtain $\mathrm{COL}(1) = B$ which is a contradiction.

The key to the last proof was that

- $(3a + 3b) + (9a + 3b) - (12a + 6b) = 0$.
- COL is defined on $(3a + 3b) + (9a + 3b) = 12a + 6b$.

For all later proofs we just give positive forbidden distances d_1, d_2, d_3 such that $d_1 + d_2 - d_3 = 0$, and conclude that the bound is $d_1 + d_2 + 1$. We abbreviate *Forbidden Distances* by FD.

We now consider $W(-ax^2 + bx; 2)$:

2) $b \geq 3a$. FD: $\{3b-3a, 6b-12a, 3b-9a\}$. $(3b-3a)+(3b-9a)-(6b-12a) = 0$.

3) $2a \leq b \leq 3a$. FD: $\{3b-3a, 6b-12a, 9a-3b\}$. $(6b-12a)+(9a-3b)-(3b-3a) = 0$.

4) $a \leq b \leq 2a$. FD: $\{3b-3a, 12a-6b, 9a-3b\}$. $(3b-3a)+(12a-6b)-(9a-3b) = 0$.

5) $0 \leq b \leq a$. FD : $\{3a-3b, 12a-6b, 9a-3b\}$. $(3a-3b)+(9a-3b)-(12a-6b) = 0$.

$\blacksquare$

Corollary 4.2. *For all $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 2) \leq 12|a| + 6|b|$.*

The bounds on $W(ax^2 + bx; 2)$ (and the others) from Theorem 4.1 hold for all a, b; however, for particular a, b better bounds can often be found. We give a class of examples.

Theorem 4.3. *Let $a \in \mathbb{N}$ with $a \geq 1$. Then $W(ax^2 + (a-1)x; 2) = 8a - 3$.*

Proof:

$W(ax^2 + (a-1)x; 2) \leq 8a - 3$.

By plugging in $x = 1, 2$ we find forbidden distances: $\{2a-1, 6a-2\}$. Since $2a-1$ is a forbidden distance, so is $3(2a-1) = 6a-3$. We will use forbidden distances $\{6a-3, 6a-2\}$.

Let $y \leq 2a-1$. Assume $\mathrm{COL}(y) = R$. Then

$$\mathrm{COL}(y) = R \rightarrow \mathrm{COL}(y + (6a-2)) = B \rightarrow \mathrm{COL}(y + (6a-2) - (6a-3)) = R.$$

To simplify, $COL(y+1) = R$. We needed $y \leq 2a-1$ since we needed $y + (6a-2) \leq 8a-3$.

Assume COL(1) $= R$. Then by applying the above we get COL(2) $= R, \ldots,$ COL($2a$) $= R$. However, since COL(1) $= R$ and $2a-1$ is a forbidden distance, COL($2a$) $= B$. This is a contradiction.

$W(ax^2 + (a-1)x; 2) \geq 8a - 3$.

We give a coloring COL of $[8a-4]$ such that, for all $x, y \in [8a-4]$ with $|x-y| \in \{2a-1, 6a-2\}$, COL(x) $\neq$ COL(y). All other forbidden distances are larger than $8a-4$ and hence irrelevant.

Here is the coloring:

1. For $1 \leq y \leq 2a-1$, COL(y) $= R$.
2. For $2a \leq y \leq 4a-2$, COL(y) $= B$.
3. For $4a-1 \leq y \leq 6a-3$, COL(y) $= R$.
4. For $6a-2 \leq y \leq 8a-4$, COL(y) $= B$.

The reader can verify that this coloring suffices. ∎

In Appendix A is a table of some exact values of $W(ax^2 + bx; 2)$.

5. $W(ax^2; 3) = 28a + 1$

In this section we will show that $W(x^2; 3) = 29$ and then $W(ax^2; 3) = 28a+1$. We first show a weaker theorem which will be a good warm-up to our work on 4-colorings in Section 8.

Theorem 5.1. $W(x^2; 3) \leq 1 + 41^2 = 1682$.

Proof:

Assume, by way of contradiction, that COL is an $(x^2; 3)$-proper coloring of $[1 + 41^2]$. Figure 1 shows some constraints on COL: COL restricted to $\{1, 17, 26, 42\}$ has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

We can assume COL(1) $= R$ and COL(17) $= B$. By looking at Figure 1 we see that COL(26) $\notin \{R, B\}$, hence COL(26) $= G$. Again by looking at Figure 1 we have that COL(42) $\notin \{B, G\}$, hence COL(42) $= R$.

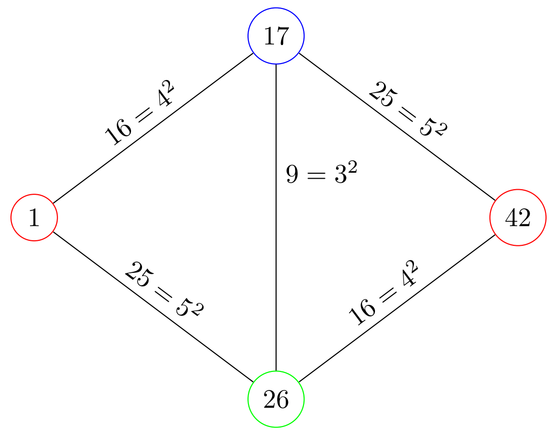

Figure 1: In any $(x^2, 3)$-proper coloring, $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$

Note that we have shown that $\mathrm{COL}(1) = \mathrm{COL}(42)$. More generally we have shown that, for all x, $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$. Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1{+}41) = \mathrm{COL}(1{+}2{\times}41) = \cdots = \mathrm{COL}(1{+}41{\times}41) = \mathrm{COL}(1{+}41^2).$$

This contradicts COL being an $(x^2; 3)$-proper coloring. ∎

The following theorem was proven by Matthew Jordan and William Gasarch.

Theorem 5.2.

1. $W(x^2; 3) = 29$.
2. *For all* $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. *This follows from Part 1 and Lemma 2.4.*

Proof:

$W(x^2; 3) \leq 29$: Assume, by way of contradiction, that there exists COL, a proper $(x^2, 3)$-coloring of $\{1, \ldots, 29\}$. Figure 2 shows some constraints on COL: COL restricted to $\{1, 10, 17, 26\}$ has to be a proper 3-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 2, $\mathrm{COL}(10) = \mathrm{COL}(17)$. By similar reasoning one can show that

$$(\forall x)[10 \leq x \leq 13 \rightarrow \mathrm{COL}(x) = \mathrm{COL}(x + 7)].$$

We refer to this fact as FORCE.

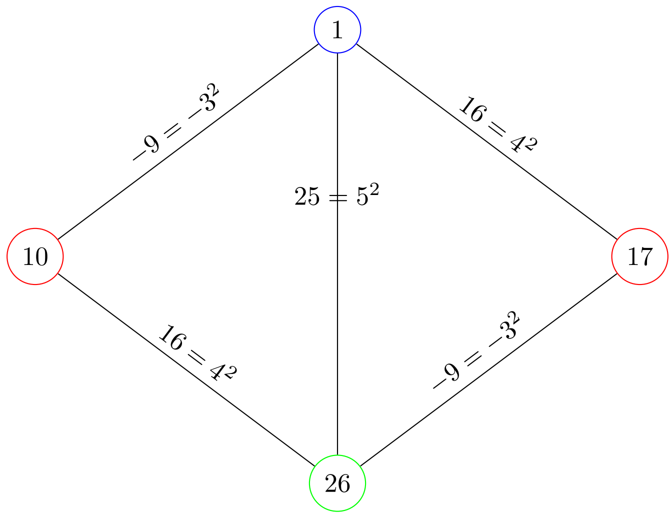

Figure 2: In any proper $(x^2, 3)$-coloring, COL(10) = COL(17)

We can assume COL(10) = R. Since $11 - 10 = 1^2$ we know that COL(10) $\neq$ COL(11), so we can assume COL(11) = B.

17: By FORCE COL(17) = COL(10) = R

18: By FORCE COL(18) = COL(11) = B.

10	11	12	13	14	15	16	17	18	19	20
R	B						R	B		

19: Since COL(10) = R and COL(18) = B, COL(19) = G.

12: By FORCE COL(12) = COL(19) = G.

10	11	12	13	14	15	16	17	18	19	20
R	B	G					R	B	G	

20: Since COL(11) = B and COL(19) = G, COL(20) = R.

13: By FORCE COL(13) = COL(20) = R.

10	11	12	13	14	15	16	17	18	19	20
R	B	G	R				R	B	G	R

Now we have that COL(17) = COL(13) = R. But $17 - 13 = 2^2$. This is a contradiction.

$W(x^2, 3) \geq 29$:

We present a proper 3-coloring:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	G	R	G	R	B	B	B	G	R	B	G	B	G

15	16	17	18	19	20	21	22	23	24	25	26	27	28
R	B	R	B	G	R	B	R	B	G	R	G	R	B

■

By Figure 2 we easily show $W(x^2; 3) \leq 68$: For $10 \leq x \leq 68$ then COL(x) = COL($x + 7$), so

$$\text{COL}(10) = \text{COL}(17) = \cdots = \text{COL}(59),$$

and note that $59 - 10 = 49 = 7^2$. This result is not as strong as $W(x^2; 3) \leq 29$; however, it has a less detailed proof.

6. Upper Bounds on $W(ax^2 + bx; 3)$

Def 6.1.

(a) A coloring of $[n]$ has *repeat distance* r if x and $x + r$ have the same color, for all $1 \leq x \leq n - r$.

(b) A coloring of $[n]$ has *repeat distance* r *under one-sided boundary condition* b if x and $x + r$ have the same color, for all $1 \leq x \leq n - r - b$.

(c) A coloring of $[n]$ has *repeat distance* r *under two-sided boundary condition* b if x and $x + r$ have the same color, for all $b < x \leq n - r - b$.

Lemma 6.2. *In any 3-coloring of* $[n]$ *with forbidden distances* $s, t, s + t$*, where* $0 < s < t$*:*

(a) $2s + t$ *is a repeat distance.*

(b) $t - s$ *is a repeat distance under two-sided boundary condition* s*.*

(c) $3s$ *is a repeat distance under one-sided boundary condition* t*.*

Proof: Let $u = s + t$.

(a) Consider a 3-coloring satisfying the conditions of the lemma. Let $1 \le x \le n - (2s + t)$. Without loss of generality, we can assume that x is R. Then $x + s$ is not R, say B, and $x + u = (x + s) + t$ cannot be R or B so it must be G. Then $(x + s) + u = (x + u) + s$ cannot be B or G so it must be R. Since x and $x + u + s$ are both R, $(x + u + s) - x = u + s = 2s + t$ is a repeat distance.

(b) Consider a 3-coloring satisfying the conditions of the lemma. Let $s < x \le n - (t - s) - s$. Without loss of generality, we can assume that x is R. Then $x - s$ is not R, say B, and $(x - s) + u = x + t$ cannot be R or B so it must be G. Then $(x - s) + t = (x + t) - s$ cannot be B or G, so it must be R. This process requires that $x - s > 0$ and $x + t \le n$. So $(x + t - s) - x = t - s$ is a repeat distance under two-sided boundary condition s.

(c) Take $2s + t$ from part (a) and subtract $t - s$ from part (b). The repeat distance is $(2s + t) - (t - s) = 3s$. There is a one-sided boundary of size $(t - s) + s = t$ from one side of part (b).

■

Lemma 6.3. *Assume* $[w]$ *has a proper 3-coloring where* s *is a forbidden distance and* r *is repeat distance under two-sided boundary condition* b*. If* $r|s$ *then*

$$w \le s + 2b + 1 \ .$$

Proof: Assume $w > s + 2b + 1$. Assume, without loss of generality, that $b + 1$ is R. Then, by Lemma 6.2b, $r + b + 1$, $2r + b + 1$, ..., $s + b + 1$ are also R, since $b + 1 > b$ and $(s + b + 1) + b = s + 2b + 1 \le n$. But s is a forbidden distance so $b + 1$ and $s + b + 1$ cannot both be R. Contradiction. ■

We use Lemma 6.3 to get upper bounds on several quadratic van der Warden numbers. For one of them we have an exact value.

Theorem 6.4.

1. *For* $a, b > 0$ *and* $a|b$, $W(ax^2 + bx; 3) \leq \frac{72b^2}{a} + 1$.
2. $W(x^2 + x; 3) = 73$.

Proof:

1) Let $p(x) = ax^2 + bx$. Let

$$x = \frac{5b}{a}, \qquad y = \frac{6b}{a}, \qquad z = \frac{8b}{a} \ .$$

Then

$$p(x) = \frac{30b^2}{a}, \qquad p(y) = \frac{42b^2}{a}, \qquad p(z) = \frac{72b^2}{a} \ .$$

Since $p(x)+p(y) = p(z)$, by Lemma 6.2b, $p(y)-p(x) = \frac{12b^2}{a}$ is a repeat distance under two-sided boundary condition $\frac{30b^2}{a}$. But $p(\frac{3b}{a}) = \frac{12b^2}{a}$ is a forbidden distance. Thus, by Lemma 6.3, $W(ax^2 + bx; 3) \leq \frac{12b^2}{a} + 2 \cdot \frac{30b^2}{a} + 1 = \frac{72b^2}{a} + 1$.
2) By Part 1 $W(x^2 + x; 3) \leq 73$. We show $W(x^2 + x; 3) \geq 73$ by giving a $(x^2 + x; 3)$-proper coloring of $\{1, \ldots, 72\}$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

∎

We now get upper bounds for $W(p(x);3)$ where $p(x) = ax^2 + bx$. Part of the proof relies on an intricate gcd calculation. We put that in the following section so that the proof flows better.

Theorem 6.5. *Let* $p(x) = ax^2 + bx$. *Then* $W(p(x);3) \leq O(|a^5 b^2|)$.

Proof: We prove the theorem for $a, b \geq 0$. The other cases are similar.

Let

$$x_0 = (2a+1)b, \qquad y_0 = (2a^2+2a+1)b, \qquad z_0 = (2a^2+2a+2)b\ .$$

Then

$$\begin{aligned} p(x_0) &= (4a^3+4a^2+3a+1)b^2 \\ p(y_0) &= (4a^5+8a^4+8a^3+6a^2+3a+1)b^2 \\ p(z_0) &= (4a^5+8a^4+12a^3+10a^2+6a+2)b^2 \end{aligned}$$

Thus $p(x_0)+p(y_0) = p(z_0)$. By Lemma 6.2b, $2p(x_0)+p(y)$ is a repeat distance, and by Lemma 6.2, $3p(x_0)$ is a repeat distance under one-sided boundary condition $p(y)$. By Lemma 7.2 we obtain that $\gcd(2p(x_0)+p(y_0), 3p(x_0)) = db^2$ for some constant $d \leq 18$. Thus there exists a linear combination over $\mathbb{Z}$ of $2p(x_0)+p(y_0)$ and $3p(x_0)$ that sums to db^2. Since both quantities are $> db^2$ the linear combination has to have one positive coefficient and one negative coefficient. We assume that the coefficient of $2p(x_0)+p(y_0)$ is positive and the coefficient of $3p(x_0)$ is negative (the other case is similar). Hence there exists $j, k \in \mathbb{N}$ such that $j(2p(x_0)+p(y_0)) - k(3p(x_0) = db^2$. By starting at 1 and adding repeat distance $2p(x_0)+p(y_0)$ j times and subtracting repeat distance $3p(x_0)$ k times, we see that db^2 is also a repeat distance. Furthermore, by interspersing the adds and subtracts so that we subtract whenever the sum is greater than $2p(x_0)+p(y_0)$, the one-sided boundary condition is $(2p(x_0)+p(y_0)) + p(y_0) = 2(p(x_0)+p(y_0))$. Thus for any integer α, αdb^2 is a repeat distance with the one-sided boundary condition $2(p(x_0)+p(y_0)) = O(a^5b^2)$. But $p(db) = ad^2b^2 + b^2d = (ad+1)db^2$ is a forbidden distance. So, $W(p(x);3) \leq p(db) + 2(p(x_0)+p(y_0)) = O(a^5b^2)$. ▮

In Appendix B is a table of some exact values of $W(ax^2 + bx; 3)$.

7. GCD calculations

Lemma 7.1. *Let $f(x) \in \mathbb{Z}[x]$ and $g(y) \in \mathbb{Z}[y]$. Assume there exists $A, B, d \in \mathbb{Z}$ such that $Af(x) + Bg(y) = d$. Then $(\forall a, b \in Z)[\gcd(f(a), g(b)) \leq d]$.*

Proof:

Since $Af(a) + Bg(b) = d$, the $\gcd(f(a), g(b))$ divides d, and hence is $\leq d$.
▮

Lemma 7.2. *Let*

$q(a, b) = (4a^3 + 4a^2 + 3a + 1)b^2$ *and* $r(a, b) = (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2$

Then, for all $a, b \in \mathbb{Z}$,

$$\gcd(2q(a, b) + r(a, b), 3q(a, b)) = db^2$$

where $d \leq 18$.

Proof: $q(a, b)$ and $r(a, b)$ factor as follows:

- $q(a, b) = (2a + 1)(2a^2 + a + 1)b^2$
- $r(a, b) = (a + 1)(2a^2 + 1)(2a^2 + 2a + 1)b^2$

Claim

1. For all $a \in \mathbb{Z}$, $\gcd(2a + 1, a + 1) = 1$.
2. For all $a \equiv 0, 2 \pmod 3$, $\gcd(2a + 1, 2a^2 + 1) = 1$.
3. For all $a \equiv 1 \pmod 3$, $\gcd(2a + 1, 2a^2 + 1) = 3$.
4. For all $a \in \mathbb{Z}$, $\gcd(2a + 1, 2a^2 + 1) = 1$.
5. For all $a \in \mathbb{Z}$, $\gcd(2a + 1, 2a^2 + 2a + 1) = 1$.
6. For all $a \equiv 0 \pmod 2$, $\gcd(2a^2 + a + 1, a + 1) = 1$.
7. For all $a \equiv 1 \pmod 2$, $\gcd(2a^2 + a + 1, a + 1) = 2$.

8. For all $a \in \mathbb{Z}$, $\gcd(2a^2 + a + 1, 2a^2 + 1) = 1$.
9. For all $a \in \mathbb{Z}$, $\gcd(2a^2 + a + 1, 2a^2 + 2a + 1) = 1$.

Proof of Claim

For all cases find a linear combination over $\mathbb{Z}$ that sums to the right hand side. In the cases where there is a condition on a such as $a \equiv 0 \pmod{3}$, first replace a as is appropriate, e.g., set $a = 3a'$. Then use Lemma 7.1 to get that gcd is $\le$ the right hand side.

If the right hand side is 1 then we are done. If the right hand side is $d \neq 1$ then its easy to show that the conditions on a make both polynomials divisible by d.

End of Proof of Claim

By the claim:

$$(\forall a, b \in \mathbb{Z})[\gcd(2a+1)(2a^2+a+1), (a+1)(2a^2+1)(2a^2+2a+1) \le 6].$$

so

$$(\forall a, b \in \mathbb{Z})[\gcd(q(a,b), r(a,b)) \le 6b^2.]$$

Hence we have, for all $a, b \in \mathbb{Z}$,

$$\begin{aligned}
\gcd(2q(a,b) + r(a,b), 3q(a,b)) &\le \gcd(3(2q(a,b) + r(a,b)) - 2(3q(a,b)), 3q(a,b)) \\
&= \gcd(3r(a,b), 3q(a,b)) \\
&= 3\gcd(r(a,b), q(a,b)) \\
&\le 18b^2.
\end{aligned}$$

$\blacksquare$

8. Upper Bounds on $W(x^2; 4)$

Recall that Figure 1 was the key to showing $W(x^2; 3) \le 1682$. We now derive parameters for a new figure that will be the key to an upper bound on $W(x^2; 4)$.

We need to find $a, b, c, d, e, f, x, y, z \in \mathbb{N}^+$ such that the following figure can be drawn:

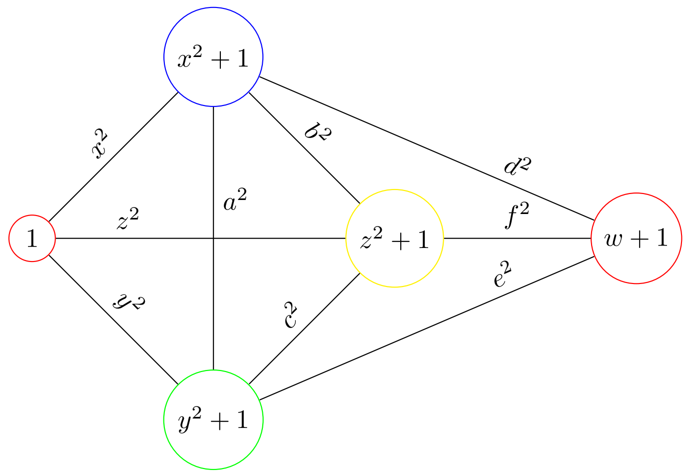

Figure 3: In any $(x^2; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + w)$

Hence we need to find solutions in $\mathbb{N}^+$ to the following system of equations:

$$
\begin{aligned}
x^2 + a^2 &= y^2 \\
x^2 + b^2 &= z^2 \\
y^2 + c^2 &= z^2 \\
x^2 + d^2 &= w \\
y^2 + e^2 &= w \\
z^2 + f^2 &= w
\end{aligned}
$$

Each equation is a Pythagorean triple, for which we have a known formula with parameters k, m, n where $m > n$, and m, n are coprime but not both odd; we can use the Farey sequence as an efficient algorithm to generate coprime pairs m, n. We used a computer program and obtained the following:

Theorem 8.1. $W(x^2; 4) \le 1 + (290{,}085{,}289)^2 = 84{,}149{,}474{,}894{,}213{,}522$

Proof:

Assume, by way of contradiction, that there exists COL, a proper $(x^2; 4)$-coloring of $[1 + (290{,}085{,}289)^2]$. Figure 4 shows some constraints on COL: COL

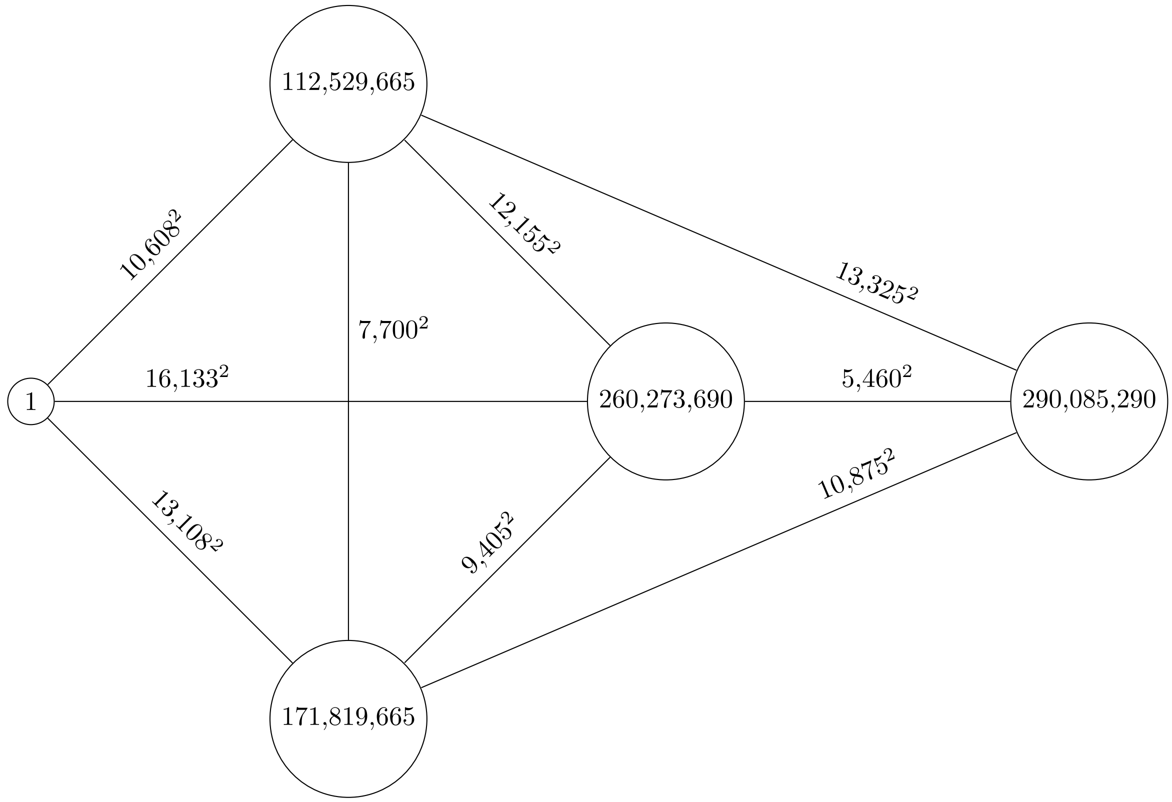

Figure 4: In any $(x^2; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + 290,085,290)$

restricted to the numbers on the vertices has to be a proper 4-coloring of the graph (no vertices that have an edge between them are the same color).

By Figure 4 we know that

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289^2).$$

More generally we have shown that, for all x,

$$\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}289^2).$$

Hence

$$\text{COL}(1) = \text{COL}(1{+}290{,}085{,}289)) = \text{COL}(1{+}2{\times}290{,}085{,}289)) = \cdots = \text{COL}(1{+}(290{,}085{,}289)^2).$$

This contradicts COL being an $(x^2; 4)$-proper coloring. $\blacksquare$

9. Upper Bounds on $W(Ax^2 + Bx; 4)$

To find upper bounds on $W(Ax^2 + Bx; 4)$ we have several overlapping equations of the form

$$(Ax^2 + Bx) + (Ay^2 + By) = (Az^2 + Bz).$$

We need a way to generate such triples (x, y, z) much like the generation of Pythagorean triples. First, we use the quadratic formula to express z in terms of x and y.

$$z = \frac{-B + \sqrt{4A^2(x^2 + y^2) + 4AB(x + y) + B^2}}{2A}$$

We rewrite as

$$4A^2(x^2 + y^2) + 4AB(x + y) + B^2 = (2Az + B)^2.$$

Simple algebra allows us to rewrite this as:

$$(2Ax + B)^2 + (2Ay + B)^2 = (2Az + B)^2 + B^2.$$

If $m = 2Ax+B$, $n = 2Ay+B$, and $k = (2Az+B)$ then we can rewrite this as $m^2 + n^2 = k^2 + B^2$. A parameterization of $m^2 + n^2 = k^2 + B^2$ would imply one for (x, y, z), and luckily this equation is easier. Using the *Bramagupta-Fibonacci identity* with $bc - ad = B$, we get:

$$(ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + B^2$$

So, with parameters a, b, c, d and some tedious algebra we get

$$x = \frac{ac - bd - B}{2A},\ y = \frac{ad + bc - B}{2A},\ z = \frac{ac + bd - B}{2A}$$

with constraints $bc - ad = B$, $ac - bd > B$, $2A | ac - bd - B$, $2A | ad + bc - B$.

Rather than searching all (a, b, c, d), we can eliminate parts of the parameter space that do not contain solutions. For fixed a and d, the first constraint implies that bc is some factorization of $ad + B$. We can pre-compute a table of factorizations and use that to cut the search space down to almost $O(n^2)$. You can see the code for this on GitHub at `https://github.com/zaprice/polyvdw`

We can get bounds for $W(Ax^2 + Bx; 4)$ with this method with rather large values of B, but only a few bounds for the more general $Ax^2 + Bx$ case; if such configurations exist, it seems the numbers involved are much larger. See Appendix C for some of the upper bounds we have. We note two things about these upper bounds:

1. The largest upper bound on $W(x^2+Bx; 4)$ that we found was when $B = 0$. Note that these are just the upper bounds we found. It is not clear how the real values compares.
2. For $W(2x^2 + Bx; 4)$ and $W(3x^2 + Bx; 4)$ the B for which we could find an upper bound seem scattered and arbitrary. For example, we were not able to find an upper bound for any of $W(2x^2 + Bx; 4)$ for $0 \leq B \leq 56$, but were able to for 57. And then not again until $B = 95$. Again, this may be a limit to our methods and not a statement about the actual values of $W(2x^2 + Bx; 4)$.

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Appendix A. Some Exact Values of $W(ax^2 + bx; 2)$

Chart of $W(p(x); 2)$ for $p(x) = ax^2 + bx$ for $0 \leq a \leq 10$ and $-10 \leq b \leq 10$. The values for $a, b \geq 0$ were obtained by using our formulas for an upper bound and then searching for a 2-coloring for the lower bound.

		a										
		0	1	2	3	4	5	6	7	8	9	10
	-10	21	1	1	9	9	1	25	11	13	17	1
	-9	19	1	9	1	7	5	7	37	15	1	23
	-8	17	1	1	7	1	7	9	13	1	21	25
	-7	15	1	7	5	5	25	11	1	19	61	29
	-6	13	1	1	1	5	9	1	17	21	25	73
	-5	11	1	5	13	7	1	15	49	25	29	31
	-4	9	1	1	5	1	13	17	23	25	33	37
	-3	7	1	3	1	11	37	19	25	31	73	41
	-2	5	1	1	9	13	19	49	29	33	39	41
	-1	3	1	7	25	17	21	27	61	37	41	47
b	0	1	5	9	13	17	21	25	29	33	37	41
	1	3	13	13	17	23	49	33	37	43	85	53
	2	5	11	25	21	25	31	33	41	45	51	97
	3	7	13	19	37	29	33	37	73	49	49	59
	4	9	17	21	27	49	37	41	47	49	57	61
	5	11	25	25	29	35	61	45	49	55	97	61
	6	13	23	25	31	37	43	73	53	57	61	65
	7	15	25	31	49	41	45	51	85	61	65	71
	8	17	29	33	39	41	49	53	59	97	69	73
	9	19	37	37	37	47	73	55	61	67	109	77
	10	21	35	49	45	49	51	57	65	69	75	121

The numbers tend to increase with increasing a and $|b|$. Some of the diagonals have patterns which likely can be used to make conjectures that are almost surely true. For example:

$$(\forall a \geq 0)[W(ax^2 - (a-1)x) = 2a+3].$$

Appendix B. Some Exact Values of $W(ax^2 + bx; 3)$

Chart of $W(p(x); 3)$ for $p(x) = ax^2 + bx$ for $0 \le a \le 5$ and $-5 \le b \le 5$. The values were obtained by computer.

		a					
		0	1	2	3	4	5
	-5	16	1	64	61	217	1
	-4	13	1	1	91	1	289
	-3	10	1	10	1	135	171
	-2	7	1	1	68	97	171
	-1	4	1	49	105	190	183
b	0	1	29	57	85	113	141
	1	4	73	76	65	156	253
	2	7	64	145	123	151	?
	3	10	37	95	217	?	?
	4	13	65	127	?	289	?
	5	16	55	?	109	?	361

Appendix C. Some Upper Bounds on $W(ax^2 + bx; 4)$

We give bounds for $W(g(x); 4)$ where g is of the form $Ax^2 + Bx$. Only bounds for coprime coefficients (A, B) are presented. Each row of the table gives g, x, y, z, w (as in Figure C.5, and the bound. We give three such tables.

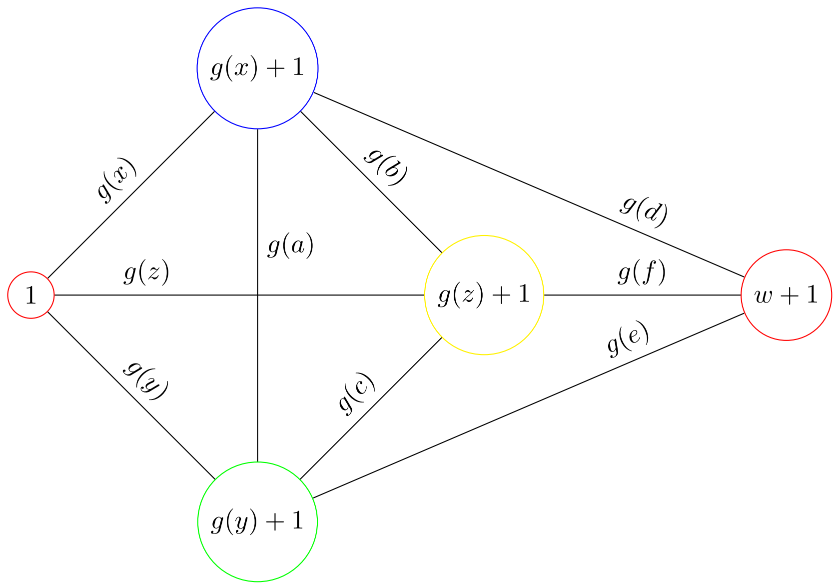

Figure C.5: In any $(g(x);4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1+w)$

Table for $x^2 + Bx$ where $0 \leq B \leq 20$.

g	x	y	z	w	$W(g(x); 4) \leq$
x^2	10,608	13,108	16,133	290,085,289	84,149,474,894,213,522
$x^2 + x$	299	302	327	113,262	12,828,393,907
$x^2 + 2x$	91	127	211	257,463	66,287,711,296
$x^2 + 3x$	35	43	53	3,308	10,952,789
$x^2 + 4x$	80	84	92	10,197	104,019,598
$x^2 + 5x$	70	81	100	11,250	126,618,751
$x^2 + 6x$	70	86	106	13,232	175,165,217
$x^2 + 7x$	638	785	923	988,338	976,818,920,611
$x^2 + 8x$	160	168	184	40,788	1,663,987,249
$x^2 + 9x$	35	37	44	3,242	10,539,743
$x^2 + 10x$	144	150	165	36,075	1,301,766,376
$x^2 + 11x$	364	472	727	1,263,252	1,595,819,511,277
$x^2 + 12x$	140	172	212	52,928	2,802,008,321
$x^2 + 13x$	119	129	143	38,016	1,445,710,465
$x^2 + 14x$	66	96	135	25,395	645,261,556
$x^2 + 15x$	120	138	215	54,364	2,956,259,957
$x^2 + 16x$	75	99	141	45,177	2,041,684,162
$x^2 + 17x$	123	165	255	232,908	54,250,095,901
$x^2 + 18x$	70	74	88	12,968	168,402,449
$x^2 + 19x$	65	66	69	6,852	47,080,093
$x^2 + 20x$	84	96	115	24,261	589,081,342

Table for $x^2 + Bx$ where $1980 \leq B \leq 2000$.

g	x	y	z	w	$W(g(x);4) \leq$
$x^2 + 1{,}980x$	1,683	2,145	2,915	25,524,829	651,567,434,640,662
$x^2 + 1{,}981x$	1,674	1,735	2,026	14,236,652	202,710,462,976,717
$x^2 + 1{,}982x$	1,248	1,495	1,731	6,882,723	47,385,517,451,716
$x^2 + 1{,}983x$	3,498	3,549	3,664	24,967,678	623,434,455,617,159
$x^2 + 1{,}984x$	860	975	2,585	12,424,497	154,392,775,905,058
$x^2 + 1{,}985x$	867	1,098	2,365	11,200,200	125,466,712,437,001
$x^2 + 1{,}986x$	1,900	2,432	2,908	19,712,552	388,623,855,480,977
$x^2 + 1{,}987x$	3,048	3,393	3,987	39,165,018	1,533,976,455,831,091
$x^2 + 1{,}988x$	508	738	1,194	6,489,996	42,132,950,192,065
$x^2 + 1{,}989x$	2,023	2,288	3,094	18,950,528	359,160,204,078,977
$x^2 + 1{,}990x$	1,364	1,610	2,100	13,163,856	173,313,300,862,177
$x^2 + 1{,}991x$	1,330	1,519	1,814	7,817,030	61,121,521,727,631
$x^2 + 1{,}992x$	975	1,065	1,871	10,120,498	102,444,639,800,021
$x^2 + 1{,}993x$	1,985	2,349	4,373	68,596,488	4,705,614,878,734,729
$x^2 + 1{,}994x$	1,246	1,350	1,716	8,551,440	73,144,177,644,961
$x^2 + 1{,}995x$	891	1,185	1,464	10,543,450	111,185,372,085,251
$x^2 + 1{,}996x$	705	995	1,793	7,390,317	54,631,536,433,222
$x^2 + 1{,}997x$	1,081	1,136	1,391	8,040,026	64,658,074,012,599
$x^2 + 1{,}998x$	1,292	1,732	3,704	39,649,768	1,572,183,322,690,289
$x^2 + 1{,}999x$	1,235	1,757	2,789	14,633,322	214,163,364,766,363
$x^2 + 2{,}000x$	184	280	984	5,592,000	31,281,648,000,001

Table for $2x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$2x^2 + 57x$	3,969	4,035	4,295	38,199,155	2,918,353,062,779,886
$2x^2 + 95x$	707	758	1,008	14,365,638	412,744,475,029,699
$2x^2 + 171x$	11,907	12,105	12,885	343,792,395	236,386,480,508,171,596
$2x^2 + 285x$	2,121	2,274	3,024	129,290,742	33,432,228,781,682,599
$2x^2 + 399x$	27,783	28,245	30,065	1,871,758,595	7,006,961,222,744,427,456
$2x^2 + 455x$	3,320	3,663	4,170	39,229,128	3,077,866,816,534,009
$2x^2 + 511x$	2,772	3,367	6,282	131,899,720	34,795,139,672,913,721
$2x^2 + 627x$	43,659	44,385	47,245	4,622,097,755	5,834,090,064,188,269,204
$2x^2 + 805x$	1,210	1,303	2,920	87,446,025	15,293,684,970,651,376
$2x^2 + 855x$	5,548	7,087	13,262	530,042,423	561,890,393,545,693,524
$2x^2 + 1{,}011x$	5,164	6,568	9,889	318,517,859	202,907,575,025,443,212
$2x^2 + 1{,}153x$	12,705	12,726	12,970	352,488,525	248,496,726,932,620,576
$2x^2 + 1{,}199x$	8,245	8,710	9,748	221,108,291	97,778,017,806,722,272
$2x^2 + 1{,}295x$	14,030	14,355	22,244	1,162,712,925	2,703,804,197,637,349,126
$2x^2 + 1{,}301x$	25,622	26,105	28,172	1,638,880,116	5,371,858,201,423,377,829
$2x^2 + 1{,}365x$	9,960	10,989	12,510	353,062,152	249,306,248,279,579,689
$2x^2 + 1{,}459x$	954	1,174	1,379	58,465,486	6,836,511,407,576,467
$2x^2 + 1{,}545x$	11,298	11,815	12,860	425,440,418	361,999,755,841,475,259
$2x^2 + 1{,}685x$	10,695	10,968	11,570	289,144,125	167,209,137,251,881,876
$2x^2 + 1{,}753x$	3,586	5,236	8,232	181,967,394	66,224,583,947,144,155
$2x^2 + 1{,}851x$	50,031	51,441	55,164	6,379,649,159	7,612,882,297,751,201,408
$2x^2 + 1{,}913x$	2,261	3,366	5,324	81,424,299	13,259,988,699,966,790

Table for $3x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$3x^2 + x$	42,273	42,660	43,375	5,738,872,934	6,570,267,294,984,419,923
$3x^2 + 143x$	13,244	13,332	13,442	554,651,696	922,915,590,942,221,777
$3x^2 + 172x$	4,452	4,712	5,189	88,862,311	23,689,546,233,099,656
$3x^2 + 200x$	1,896	2,204	5,004	115,177,723	39,797,746,661,938,788
$3x^2 + 235x$	11,155	11,270	11,610	583,594,418	1,021,747,471,306,964,403
$3x^2 + 274x$	9,322	11,610	16,903	1,125,018,929	3,797,003,080,080,107,670
$3x^2 + 344x$	8,904	9,424	10,378	355,449,244	379,032,617,455,054,545
$3x^2 + 361x$	3,540	4,658	7,703	397,333,094	473,620,906,200,085,443
$3x^2 + 400x$	3,792	4,408	10,008	460,710,892	636,763,762,306,663,793
$3x^2 + 407x$	2,806	3,401	6,131	122,898,626	45,312,266,837,804,411
$3x^2 + 412x$	2,077	2,829	5,839	392,773,686	462,813,667,064,838,421
$3x^2 + 520x$	7,616	9,244	12,716	515,261,395	796,483,183,467,963,476
$3x^2 + 556x$	9,400	9,408	9,451	273,674,799	224,693,838,986,259,448
$3x^2 + 592x$	15,744	16,472	17,944	994,061,387	2,964,474,711,857,432,412
$3x^2 + 643x$	50,932	51,357	52,351	8,273,167,696	2,421,731,687,255,606,001
$3x^2 + 688x$	17,808	18,848	20,756	1,421,796,976	6,064,520,901,084,553,217
$3x^2 + 725x$	3,172	3,185	3,278	34,869,750	3,647,723,675,756,251
$3x^2 + 728x$	16,744	17,360	18,928	1,174,742,491	4,140,060,615,695,188,692
$3x^2 + 797x$	2,847	3,082	3,524	148,907,272	66,520,245,642,541,737
$3x^2 + 814x$	5,612	6,802	12,262	491,594,504	724,995,869,246,944,305
$3x^2 + 932x$	1,820	2,229	2,799	37,745,311	4,274,160,686,090,016
$3x^2 + 1{,}085x$	1,190	1,344	1,540	10,401,450	324,581,771,880,751
$3x^2 + 1{,}087x$	9,800	9,909	11,434	604,108,526	1,094,841,990,223,645,791
$3x^2 + 1{,}112x$	18,800	18,816	18,902	1,094,699,196	3,595,100,206,474,645,201