1. A Non-HS Idea for $W(x^2; c)$

Theorem ?? gave an enormous upper bound on $W(x^2; 4)$. The proof was found by a computer program; however, it is a HS proof and human-verifiable. Four colors seems to be at the limit of what computers can find. That is, we have been unable to use a program to find a human-verifiable proof for a bound on $W(x^2; 5)$.

There is another possible approach. Usually a HS proof gives better bounds than a proof that uses advanced mathematics. However, our HS proof for $W(x^2; 4)$ gives such a large bound that its possible the advanced proofs, if looked at more carefully, will yield better bounds on $W(x^2; 4)$. It’s also possible they will yield reasonable bounds for $W(x^2; c)$ for small value of $c$ such as $c = 5$.

We summarize the literature on the following problem: find the smallest possible function $a(n)$ such that, for large $n$, any $X \subseteq \{1, \ldots, n\}$ of density $\Omega(a(n))$ (that is, $|X| \geq \Omega(\frac{a(n)}{n})$) has two numbers that are a square apart. It is easy to see that, for large $n$, $W(x^2; O(\frac{1}{a(n)})) \leq n$ (which can be used to get a bound no $W(x^2; c)$). The proofs are asymptotic and not HS; however, it is possible the can be modified to give actual upper bounds on $W(x^2; c)$.

<table>
<thead>
<tr>
<th>$a(n)$</th>
<th>Reference</th>
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<tbody>
<tr>
<td>1</td>
<td>Furstenberg [7]</td>
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<td>1</td>
<td>Lyall [18] (simpler proof but not HS)</td>
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<tr>
<td>$\left(\frac{\log \log n}{\log n}\right)^{2/3}$</td>
<td>Sárközy [8]</td>
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<td>$\left(\frac{\log \log n}{\log n}\right)^{1/3}$</td>
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<td>$\left(\frac{\log n}{\log \log n}\right)^{-c}$</td>
<td>Green [19]</td>
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<td>$\frac{1}{(\log n)^{1 + \frac{c}{\log \log n}}}$</td>
<td>Bloom &amp; Maynard [20]</td>
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</table>

References


[12] S. Peluse, Bounds for sets with no polynomial progressions, Forum of Mathematics Pi 8 (2020) e16,  

[13] S. Peluse, S. Prendville, Quantitative bounds in the non-linear Roth theorem,  


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