f(m,3) to f(m,9)

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Muffins divided into **pieces**

Students get **shares**

FC thm:

$$f(m,s) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}$$

We get our upper bounds in one of three ways:

1. From the FC thm with the min part and the upper bound is not $\frac{1}{3}$. This is most of our cases so we do not comment on it.
2. From the FC thm with upper bound from the $\frac{1}{3}$. We call these $\frac{1}{3}$**-exceptions** (There is only one case where the min part equals $\frac{1}{3}$, thats $f(4,3)$.) In this case the procedures we give below are not optimal so we give another one.
3. Using a technique that is not FC. These are called **exceptions**. We note them and state what techniques do give the upper bound and give the procedure. There should be a reason our formulas do not work (e.g., they involve giving someone -1 muffins).

1 f(m,3)

1.1 f(3k+1,3)

$f(3k+1,3) = \frac{3k-1}{6k}$ for $k \geq 1$:

1. $2k$ muffins divided $(\frac{3k-1}{6k}, \frac{3k+1}{6k})$
2. $k+1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. Give 1 student $[2k : \frac{3k+1}{6k}]$
4. Give 2 students $[k : \frac{3k-1}{6k}]$ and $[k+1 : \frac{1}{2}]$

1.2 f(3k+2,3)

$f(3k+2,3) = \frac{3k+2}{6k+6}$ for $k \geq 1$:

1. $2k+2$ muffins divided $(\frac{3k+2}{6k+6}, \frac{3k+4}{6k+6})$
2. k muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. Give 1 student $[2k+2 : \frac{3k+2}{6k+6}]$
4. Give 2 students $[k+1 : \frac{3k+4}{6k+6}]$ and $[k : \frac{1}{2}]$

2 f(m,4)

2.1 f(4k+1,4)

$f(4k+1,4) = \frac{4k-1}{8k}$ for $k \geq 1$:

1. $4k$ muffins divided $(\frac{4k-1}{8k}, \frac{4k+1}{8k})$
2. 1 muffin divided $(\frac{1}{2}, \frac{1}{2})$
3. Give 2 students $2k$ shares of size $\frac{4k+1}{8k}$
4. Give 2 students 1 share of size $\frac{1}{2}$ and $2k$ shares of size $\frac{4k-1}{8k}$

2.2 f(4k+3,4)

$f(4k+3,4) = \frac{4k+1}{8k+4}$ for $k \geq 1$:

1. $4k+2$ muffins divided $(\frac{4k+1}{8k+4}, \frac{4k+3}{8k+4})$
2. 1 muffin divided $(\frac{1}{2}, \frac{1}{2})$
3. Give 2 students $2k+1$ shares of size $\frac{4k+3}{8k+4}$
4. Give 2 students $2k+1$ shares of size $\frac{4k+1}{8k+4}$ and 1 share of size $\frac{1}{2}$

3 f(m,5)

3.1 f(5k+1,5)

$f(5k+1,5) = \frac{5k+1}{10k+5}$ for $k \geq 1$, $k \neq 2$:

If $k \equiv 0 \pmod 3$:

1. $4k+2$ muffins divided $(\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5})$
2. 2 muffins divided $(\frac{5k+2}{10k+5}, \frac{5k+3}{10k+5})$
3. $k-3$ muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{5k+1}{10k+5}]$
5. Give 2 students $[\frac{4k+3}{3} : \frac{5k+4}{10k+5}]$ and $[1 : \frac{5k+2}{10k+5}]$ and $[\frac{k-3}{3} : 1]$
6. Give 1 student $[\frac{4k}{3} : \frac{5k+4}{10k+5}]$ and $[2 : \frac{5k+3}{10k+5}]$ and $[\frac{k-3}{3} : 1]$

If $k \equiv 1 \pmod 3$:

1. $4k+2$ muffins divided $[\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5}]$
2. $k-1$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{5k+1}{10k+5}]$
4. Give 3 students $[\frac{4k+2}{3} : \frac{5k+4}{10k+5}]$ and $[\frac{k-1}{3} : 1]$

If $k \equiv 2 \pmod 3$ and $k \neq 2$:

1. $4k+2$ muffins divided $[\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5}]$
2. 2 muffins divided $[\frac{5k+2}{10k+5}, \frac{5k+3}{10k+5}]$
3. $k-3$ muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{5k+1}{10k+5}]$
5. Give 2 students $[\frac{4k+1}{3} : \frac{5k+4}{10k+5}]$ and $[1 : \frac{5k+3}{10k+5}]$ and $[\frac{k-2}{3} : 1]$
6. Give 1 student $[\frac{4k+4}{3} : \frac{5k+4}{10k+5}]$ and $[2 : \frac{5k+2}{10k+5}]$ and $[\frac{k-5}{3} : 1]$ (Since $k \equiv 2 \pmod 3$, $k = 2, 5, 8, \ldots$. When $k = 2$ step 6 involves -3 shares which is impossible.)

EXCEPTION: $f(11,5) = \frac{13}{30}$. Upper bound by HALF,INT,MID

1. 8 muffins divided $[\frac{13}{30}, \frac{17}{30}]$
2. 2 muffins divided $[\frac{14}{30}, \frac{16}{30}]$
3. 1 muffins divided $[\frac{15}{30}, \frac{15}{30}]$
4. Give 2 students $[\frac{14}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}]$
5. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$

3.2 f(5k+2,5)

$f(5k+2,5) = \frac{5k-2}{10k}$ for $k \geq 2$:

If k is odd and $k \geq 2$:

1. $2k$ muffins divided $(\frac{5k+2}{10k}, \frac{5k-2}{10k})$
2. 2 muffins divided $(\frac{5k-1}{10k}, \frac{5k+1}{10k})$
3. $3k$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. Give 1 student $[2k : \frac{5k+2}{10k}]$
5. Give 2 students $[\frac{k+1}{2} : \frac{5k-2}{10k}]$, $[1 : \frac{5k+1}{10k}]$ and $[\frac{3k-1}{2} : \frac{1}{2}]$
6. Give 2 student $[\frac{k-1}{2} : \frac{5k-2}{10k}]$ and $[1 : \frac{5k-1}{10k}]$ and $[\frac{3k+1}{2} : \frac{1}{2}]$

$\frac{1}{3}$-EXCEPTION: $f(7,5) = \frac{1}{3}$. Upper bound by HALF, INT, MID

1. 3 muffins divided $[\frac{5}{15}, \frac{5}{15}, \frac{5}{15}]$
2. 2 muffins divided $[\frac{7}{15}, \frac{8}{15}]$
3. 2 muffins divided $[\frac{6}{15}, \frac{9}{15}]$
4. Give 2 students $[\frac{5}{15}, \frac{5}{15}, \frac{5}{15}, \frac{6}{15}]$
5. Give 1 students $[\frac{5}{15}, \frac{8}{15}, \frac{8}{15}]$
6. Give 2 students $[\frac{5}{15}, \frac{7}{15}, \frac{9}{15}]$

If k is even:

1. $2k$ muffins divided $(\frac{5k-2}{10k}, \frac{5k+2}{10k})$
2. $3k+2$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. Give 1 student $[2k : \frac{5k+2}{10k}]$
4. Give 4 students $[\frac{k}{2} : \frac{5k-2}{10k}]$ and $[\frac{3k+2}{2} : \frac{1}{2}]$

3.3 f(5k+3,5)

$f(5k+3,5) = \frac{5k+3}{10k+10}$ for $k \geq 1$:

If k is odd:

1. $4k+4$ muffins divided $(\frac{5k+3}{10k+10}, \frac{5k+7}{10k+10})$
2. $k-1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. 1 student gets $[2k+2 : \frac{5k+3}{10k+10}]$
4. 4 students get $[\frac{k+1}{2} : \frac{5k+3}{10k+10}]$ and $[k+1 : \frac{5k+7}{10k+10}]$ and $[\frac{k-1}{2} : \frac{1}{2}]$

If k is even:

1. $2k+2$ muffins divided $(\frac{5k+3}{10k+10}, \frac{5k+7}{10k+10})$
2. 2 muffins divided $(\frac{5k+4}{10k+10}, \frac{5k+6}{10k+10})$
3. $3k-1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k+2 : \frac{5k+3}{10k+10}]$
5. 2 students get $[\frac{k+2}{2} : \frac{5k+7}{10k+10}]$ and $[1:\frac{5k+4}{10k+10}]$ and $[\frac{3k-2}{2} : \frac{1}{2}]$
6. 2 students get $[\frac{k}{2} : \frac{5k+7}{10k+10}]$ and $[1:\frac{5k+6}{10k+10}]$ and $[\frac{3k}{2} : \frac{1}{2}]$

3.4 f(5k+4,5)

$f(5k+4,5) = \frac{5k+1}{10k+5}$ for $k \geq 1$:

If $k \equiv 0 \pmod 3$:

1. $4k+2$ muffins divided $(\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5})$
2. 2 muffins divided $(\frac{5k+2}{10k+5}, \frac{5k+3}{10k+5})$
3. k muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{5k+4}{10k+5}]$
5. Give 2 students $[\frac{4k+3}{3} : \frac{5k+1}{10k+5}]$ and $[1: \frac{5k+3}{10k+5}]$ and $[\frac{k}{3}:1]$
6. Give 1 student $[\frac{4k}{3} : \frac{5k+1}{10k+5}]$ and $[2: \frac{5k+2}{10k+5}]$ and $[\frac{k}{3}:1]$

If $k \equiv 1 \pmod 3$:

1. $4k+2$ muffins divided $(\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5})$
2. $k+2$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{5k+4}{10k+5}]$
4. Give 3 students $[\frac{4k+2}{3} : \frac{5k+1}{10k+5}]$ and $[\frac{k+2}{3} : 1]$

If $k \equiv 2 \pmod 3$:

1. $4k+2$ muffins divided $(\frac{5k+1}{10k+5}, \frac{5k+4}{10k+5})$
2. 2 muffins divided $(\frac{5k+2}{10k+5}, \frac{5k+3}{10k+5})$
3. k muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{5k+4}{10k+5}]$
5. Give 2 students $[\frac{4k+1}{3} : \frac{5k+1}{10k+5}]$ and $[1 : \frac{5k+2}{10k+5}]$ and $[\frac{k+1}{3} : 1]$
6. Give 1 student $[\frac{4k+4}{3} : \frac{5k+1}{10k+5}]$ and $[2 : \frac{5k+3}{10k+5}]$ and $[\frac{k-2}{3} : 1]$

4 f(m,6)

4.1 f(6k+1,6)

$f(6k+1,6) = \frac{6k+1}{12k+6}$ for $k \neq 1$:

1. $4k+2$ muffins divided $(\frac{6k+1}{12k+6}, \frac{6k+5}{12k+6})$

2. 2 muffins divided $(\frac{6k+2}{12k+6}, \frac{6k+4}{12k+6})$

3. $2k-3$ muffins divided $(\frac{1}{2}, \frac{1}{2})$

4. Give 2 students $[2k+1 : \frac{6k+1}{12k+6}]$

5. Give 2 students $[k+1 : \frac{6k+5}{12k+6}]$ and $[1 : \frac{6k+2}{12k+6}]$ and $[k-2 : \frac{1}{2}]$ (If $k = 1$ then this gives a $k-2 = -1$ muffins which is impossible.)

6. Give 2 students $[k : \frac{6k+5}{12k+6}]$ and $[1 : \frac{6k+4}{12k+6}]$ and $[k-1 : \frac{1}{2}]$

EXCEPTION: $f(7,6) = \frac{1}{3}$. Upper bound by HALF, INT, MID, EBM

1. 4 muffins divided $[\frac{2}{6}, \frac{2}{6}, \frac{2}{6}]$

2. 3 muffins divided $[\frac{3}{6}, \frac{3}{6}]$

3. Give 6 students $[\frac{3}{6}, \frac{2}{6}, \frac{2}{6}]$

4.2 f(6k+5,6)

$f(6k+5,6) = \frac{6k+1}{12k+6}$ for $k \geq 1$:

1. $4k+2$ muffins divided $(\frac{6k+1}{12k+6}, \frac{6k+5}{12k+6})$
2. 2 muffins divided $(\frac{6k+2}{12k+6}, \frac{6k+4}{12k+6})$
3. $2k+1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. Give 2 students $[2k+1 : \frac{6k+5}{12k+6}]$
5. Give 2 students $[k+1 : \frac{6k+1}{12k+6}]$ and $[1 : \frac{6k+4}{12k+6}]$ and $[k : \frac{1}{2}]$
6. Give 2 students $[k : \frac{6k+1}{12k+6}]$ and $[1 : \frac{6k+2}{12k+6}]$ and $[k+1 : \frac{1}{2}]$

5 f(m,7)

5.1 f(7k,7)

Give every student k whole muffins.

5.2 f(7k+1,7)

$f(7k+1,7) = \frac{7k+1}{14k+7}$ for $k \geq 2$:

If $k \equiv 0 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-7$ muffins UNDIVIDED
3. 6 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+1}{14k+7}]$
5. Give 2 students $[\frac{4k+5}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-10}{5} : 1]$ and $[3 : \frac{7k+3}{14k+7}]$
6. Give 3 students $[\frac{4k}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-5}{5} : 1]$ and $[2 : \frac{7k+4}{14k+7}]$

If $k \equiv 1 \pmod 5$ and $k \geq 2$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-5$ muffins UNDIVIDED
3. 4 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+1}{14k+7}]$
5. Give 4 students $[\frac{4k+1}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-3}{5} : 1]$ and $[1 : \frac{7k+4}{14k+7}]$
6. Give 1 student $[\frac{4k+6}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-13}{5} : 1]$ and $[4 : \frac{7k+3}{14k+7}]$ (Since $k \equiv 1 \pmod 5$, $k = 1, 6, 11, \ldots$. When $k = 1$ step 6 involves $3k - 13 = -10$, so this is impossible.)

EXCEPTION: $f(8,7) = \frac{5}{14}$. Upper bound by HALF, INT, MID, EBM

1. 4 muffins divided $(\frac{5}{14}, \frac{9}{14})$
2. 2 muffins divided $(\frac{6}{14}, \frac{8}{14})$
3. 2 muffins divided $(\frac{7}{14}, \frac{7}{14})$
4. Give 2 students $[\frac{5}{14}, \frac{5}{14}, \frac{6}{14}]$
5. Give 4 students $[\frac{7}{14}, \frac{9}{14}]$
6. Give 1 students $[\frac{8}{14}, \frac{8}{14}]$

If $k \equiv 2 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-1$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{7k+1}{14k+7}]$
4. Give 5 students $[\frac{4k+2}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-1}{5} : 1]$

If $k \equiv 3 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-5$ muffins UNDIVIDED
3. 4 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+1}{14k+7}]$
5. Give 4 students $[\frac{4k+3}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-4}{5} : 1]$ and $[1 : \frac{7k+3}{14k+7}]$
6. Give 1 student $[\frac{4k-2}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-9}{5} : 1]$ and $[4 : \frac{7k+4}{14k+7}]$

If $k \equiv 4 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-7$ muffins UNDIVIDED
3. 6 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+1}{14k+7}]$
5. Give 2 students $[\frac{4k-1}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-7}{5} : 1]$ and $[3 : \frac{7k+4}{14k+7}]$
6. Give 3 students $[\frac{4k+4}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-7}{5} : 1]$ and $[2 : \frac{7k+3}{14k+7}]$

5.3 f(7k+2,7)

$f(7k+2,7) = \frac{7k-2}{14k}$ for $k \geq 1$:

If k is odd:

1. $6k$ muffins divided $(\frac{7k-2}{14k}, \frac{7k+2}{14k})$
2. 2 muffins divided $(\frac{7k-1}{14k}, \frac{7k+1}{14k})$
3. k muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 3 students get $[2k : \frac{7k+2}{14k}]$
5. 2 students get $[\frac{3k+1}{2} : \frac{7k-2}{14k}]$ and $[1 : \frac{7k+1}{14k}]$ and $[\frac{k-1}{2} : \frac{1}{2}]$
6. 2 students get $[\frac{3k-1}{2} : \frac{7k-2}{14k}]$ and $[1 : \frac{7k-1}{14k}]$ and $[\frac{k+1}{2} : \frac{1}{2}]$

If k is even:

1. $6k$ muffins divided $(\frac{7k-2}{14k}, \frac{7k+2}{14k})$
2. $k+2$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. 3 students get $[2k : \frac{7k+2}{14k}]$
4. 4 students get $[\frac{3k}{2} : \frac{7k-2}{14k}]$ and $[\frac{k+2}{2} : \frac{1}{2}]$

5.4 f(7k+3,7)

$f(7k+3,7) = \frac{7k-3}{14k}$ for $k \neq 1$:

If k is odd and $k \neq 1$:

1. $2k$ muffins divided $(\frac{7k-3}{14k}, \frac{7k+3}{14k})$
2. $2k$ muffins divided $(\frac{7k-1}{14k}, \frac{7k+1}{14k})$
3. $3k+3$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k : \frac{7k+3}{14k}]$
5. 2 students get $[\frac{k+1}{2} : \frac{7k-3}{14k}]$ and $[\frac{k+3}{2} : \frac{7k+1}{14k}]$ and $[k-1 : \frac{1}{2}]$
6. 2 students get $[\frac{k-1}{2} : \frac{7k-3}{14k}]$ and $[\frac{k-3}{2} : \frac{7k+1}{14k}]$ and $[k+3 : \frac{1}{2}]$
7. 2 students get $[k : \frac{7k-1}{14k}]$ and $[k+1 : \frac{1}{2}]$

If $k = 1$ then the above procedure gives $f(10,7) \geq \frac{2}{7}$ which is true but we can do better.

$\frac{1}{3}$-EXCEPTION: $f(10,7) = \frac{1}{3}$. Upper bound by FC, EBM

1. 4 muffins divided $(\frac{7}{21}, \frac{7}{21}, \frac{7}{21})$
2. 3 muffins divided $(\frac{9}{21}, \frac{12}{21})$
3. 3 muffins divided $(\frac{10}{21}, \frac{11}{21})$
4. Give 3 students $[\frac{7}{21}, \frac{7}{21}, \frac{7}{21}, \frac{9}{21}]$
5. Give 3 students $[\frac{7}{21}, \frac{11}{21}, \frac{12}{21}]$
6. Give 1 students $[\frac{10}{21}, \frac{10}{21}, \frac{10}{21}]$

If k is even:

1. $2k$ muffins divided $(\frac{7k-3}{14k}, \frac{7k+3}{14k})$
2. $3k+3$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. $2k$ muffins divided $(\frac{7k-1}{14k}, \frac{7k+1}{14k})$
4. 1 student gets $[2k : \frac{7k+3}{14k}]$
5. 4 students get $[\frac{k}{2} : \frac{7k-3}{14k}]$ and $[k+1 : \frac{1}{2}]$ and $[\frac{k}{2} : \frac{7k+1}{14k}]$
6. 2 students get $[k : \frac{7k-1}{14k}]$ and $[k+1 : \frac{1}{2}]$

5.5 f(7k+4,7)

$f(7k+4,7) = \frac{7k+4}{14k+14}$ for $k \geq 1$:

If k is odd:

1. $2k+2$ muffins divided $(\frac{7k+4}{14k+14}, \frac{7k+10}{14k+14})$
2. $3k$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. $2k+2$ muffins divided $(\frac{7k+6}{14k+14}, \frac{7k+8}{14k+14})$
4. 1 student gets $[2k+2 : \frac{7k+4}{14k+14}]$
5. 4 students get $[\frac{k+1}{2} : \frac{7k+10}{14k+14}]$ and $[k : \frac{1}{2}]$ and $[\frac{k+1}{2} : \frac{7k+6}{14k+14}]$
6. 2 students get $[k+1 : \frac{7k+8}{14k+14}]$ and $[k : \frac{1}{2}]$

If k is even:

1. $2k+2$ muffins divided $(\frac{7k+4}{14k+14}, \frac{7k+10}{14k+14})$
2. $2k+2$ muffins divided $(\frac{7k+6}{14k+14}, \frac{7k+8}{14+14})$
3. $3k$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k+2 : \frac{7k+4}{14k+14}]$
5. 2 students get $[\frac{k+2}{2} : \frac{7k+10}{14k+14}]$ and $[\frac{k+4}{2} : \frac{7k+6}{14k+14}]$ and $[k-2 : \frac{1}{2}]$
6. 2 students get $[\frac{k}{2} : \frac{7k+10}{14k+14}]$ and $[\frac{k-2}{2} : \frac{7k+6}{14k+14}]$ and $[k+2 : \frac{1}{2}]$
7. 2 students get $[k+1 : \frac{7k+8}{14k+14}]$ and $[k : \frac{1}{2}]$

5.6 f(7k+5,7)

$f(7k+5,7) = \frac{7k+5}{14k+14}$ for $k = 1$ and $k \geq 3$:

If k is odd:

1. $6k+6$ muffins divided $(\frac{7k+5}{14k+14}, \frac{7k+9}{14k+14})$
2. $k-1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
3. 3 students get $[2k+2 : \frac{7k+5}{14k+14}]$
4. 4 students get $[\frac{3k+3}{2} : \frac{7k+9}{14k+14}]$ and $[\frac{k-1}{2} : \frac{1}{2}]$

If k is even and $k \geq 4$:

1. $6k+6$ muffins divided $(\frac{7k+5}{14k+14}, \frac{7k+9}{14k+14})$
2. 2 muffins divided $(\frac{7k+6}{14k+14}, \frac{7k+8}{14k+14})$
3. $k-3$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 3 students get $[2k+2 : \frac{7k+5}{14k+14}]$
5. 2 students get $[\frac{3k+4}{2} : \frac{7k+9}{14k+14}]$ and $[1 : \frac{7k+6}{14k+14}]$ and $[\frac{k-4}{2} : \frac{1}{2}]$ (Since k is even, $k = 2, 4, 6, \ldots$. When $k = 2$ step 5 involves $k-4 = -2$ shares which is impossible.)
6. 2 students get $[\frac{3k+2}{2} : \frac{7k+9}{14k+14}]$ and $[1 : \frac{7k+8}{14k+14}]$ and $[\frac{k-2}{2} : \frac{1}{2}]$

EXCEPTION: $f(19,7) = \frac{25}{56}$. Upper bound by HALF, INT, MID.

1. 12 muffins divided $(\frac{25}{56}, \frac{31}{56})$
2. 6 muffins divided $(\frac{26}{56}, \frac{30}{56})$
3. 1 muffins divided $(\frac{28}{56}, \frac{28}{56})$
4. Give 3 students $[\frac{25}{56}, \frac{25}{56}, \frac{25}{56}, \frac{25}{56}, \frac{26}{56}]$
5. Give 2 students $[\frac{28}{56}, \frac{31}{56}, \frac{31}{56}, \frac{31}{56}, \frac{31}{56}]$

5.7 f(7k+6,7)

$f(7k+6,7) = \frac{7k+1}{14k+7}$ for $k \geq 1$:

If $k \equiv 0 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-2$ muffins UNDIVIDED
3. 6 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+6}{14k+7}]$
5. Give 2 students $[\frac{4k+5}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k-5}{5} : 1]$ and $[3 : \frac{7k+4}{14k+7}]$
6. Give 3 students $[\frac{4k}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k}{5} : 1]$ and $[2 : \frac{7k+3}{14k+7}]$

If $k \equiv 1 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k$ muffins UNDIVIDED
3. 4 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+6}{14k+7}]$
5. Give 4 students $[\frac{4k+1}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k+2}{5} : 1]$ and $[1 : \frac{7k+3}{14k+7}]$
6. Give 1 student $[\frac{4k+6}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k-8}{5} : 1]$ and $[4 : \frac{7k+4}{14k+7}]$

If $k \equiv 2 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k+4$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{7k+6}{14k+7}]$
4. Give 5 students $[\frac{4k+2}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k+4}{5} : 1]$

If $k \equiv 3 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k$ muffins UNDIVIDED
3. 4 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+6}{14k+7}]$
5. Give 4 students $[\frac{4k+3}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k+1}{5} : 1]$ and $[1 : \frac{7k+4}{14k+7}]$
6. Give 1 student $[\frac{4k-2}{5} : \frac{7k+6}{14k+7}]$ and $[\frac{3k-4}{5} : 1]$ and $[4 : \frac{7k+3}{14k+7}]$

If $k \equiv 4 \pmod 5$:

1. $4k+2$ muffins divided $(\frac{7k+1}{14k+7}, \frac{7k+6}{14k+7})$
2. $3k-2$ muffins UNDIVIDED
3. 6 muffins divided $(\frac{7k+3}{14k+7}, \frac{7k+4}{14k+7})$
4. Give 2 students $[2k+1 : \frac{7k+6}{14k+7}]$
5. Give 2 students $[\frac{4k-1}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k-2}{5} : 1]$ and $[3 : \frac{7k+3}{14k+7}]$
6. Give 3 students $[\frac{4k+4}{5} : \frac{7k+1}{14k+7}]$ and $[\frac{3k-2}{5} : 1]$ and $[2 : \frac{7k+4}{14k+7}]$

6 f(m,8)

6.1 f(8k+1,8)

$f(8k+1,8) = \frac{8k+1}{16k+8}$.

1. $4k+2$ muffins divided $(\frac{8k+1}{16k+8}, \frac{8k+7}{16k+8})$
2. $4k-4$ muffins divided $(\frac{8k+3}{16k+8}, \frac{8k+5}{16k+8})$
3. 3 muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. Give 2 students $[2k+1 : \frac{8k+1}{16k+8}]$
5. Give 4 students $[k-1 : \frac{8k+3}{16k+8}]$ and $[k : \frac{8k+7}{16k+8}]$ and $[1 : \frac{1}{2}]$
6. Give 2 students $[2k-2 : \frac{8k+5}{16k+8}]$ and $[1 : \frac{8k+7}{16k+8}]$ and $[1 : \frac{1}{2}]$

6.2 f(8k+3,8)

$f(8k+3,8) = \frac{8k-3}{16k}$ for $k \neq 1$:

1. $4k$ muffins divided $(\frac{8k-3}{16k}, \frac{8k+3}{16k})$
2. $4k-8$ muffins divided $(\frac{8k-1}{16k}, \frac{8k+1}{16k})$
3. 4 muffins divided $(\frac{8k-2}{16k}, \frac{8k+2}{16k})$
4. 7 muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. Give 2 students $[2k : \frac{8k+3}{16k}]$
6. Give 4 students $[k : \frac{8k-3}{16k}]$ and $[k-2 : \frac{8k+1}{16k}]$ and $[1 : \frac{8k+2}{16k}]$ and $[2 : \frac{1}{2}]$
7. Give 2 students $[2 : \frac{8k-2}{16k}]$ and $[2k-4 : \frac{8k-1}{16k}]$ and $[3 : \frac{1}{2}]$

If $k = 1$ then line 6 says to give a student $k-2 = -1$ shares and Line 7 says to give a student $2k-4 = -2$ shares. Hence the procedure does not work when $k = 1$. The formula gives the bound $f(11,8) = \frac{5}{16}$. Actually we can do better.

$\frac{1}{3}$-EXCEPTION: $f(11,8) = \frac{1}{3}$. Upper bound by FC, EBM.

1. 2 muffins divided $(\frac{8}{24}, \frac{8}{24}\frac{8}{24})$
2. 6 muffins divided $(\frac{11}{24}, \frac{13}{24})$
3. 3 muffins divided $(\frac{12}{24}, \frac{12}{24})$
4. Give 2 students $[\frac{11}{24}, \frac{11}{24}, \frac{11}{24}]$
5. Give 6 students $[\frac{8}{24}, \frac{12}{24}, \frac{11}{24}]$

6.3 f(8k+5,8)

$f(8k+5,8) = \frac{8k+5}{16k+16}$ for $k \geq 1$:

1. $4k+4$ muffins divided $(\frac{8k+5}{16k+16}, \frac{8k+11}{16k+16})$
2. $4k-4$ muffins divided $(\frac{8k+7}{16k+16}, \frac{8k+9}{16k+16})$
3. 4 muffins divided $(\frac{8k+6}{16k+16}, \frac{8k+10}{16k+16})$
4. 1 muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. Give 2 students $[2k+2 : \frac{8k+5}{16k+16}]$
6. Give 4 students $[k+1 : \frac{8k+11}{16k+16}]$ and $[k-1 : \frac{8k+7}{16k+16}]$ and $[1 : \frac{8k+6}{16k+16}]$
7. Give 2 students $[2 : \frac{8k+10}{16k+16}]$ and $[2k-2 : \frac{8k+9}{16k+16}]$ and $[1 : \frac{1}{2}]$

6.4 f(8k+7,8)

$f(8k+7,8) = \frac{8k+1}{16k+8}$ for $k \geq 1$:

1. $4k+2$ muffins divided $(\frac{8k+1}{16k+8}, \frac{8k+7}{16k+8})$
2. $4k-4$ muffins divided $(\frac{8k+3}{16k+8}, \frac{8k+5}{16k+8})$
3. 9 muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. Give 2 students $[2k+1 : \frac{8k+7}{16k+8}]$
5. Give 4 students $[k-1 : \frac{8k+5}{16k+8}]$ and $[k : \frac{8k+1}{16k+8}]$ and $[3 : \frac{1}{2}]$
6. Give 2 students $[2k-2 : \frac{8k+3}{16k+8}]$ and $[1 : \frac{8k+1}{16k+8}]$ and $[3 : \frac{1}{2}]$

7 f(m,9)

7.1 f(9k+1,9)

$f(9k+1, 9) = \frac{9k+1}{18k+9}$ for $k \geq 2$:

If $k \equiv 0 \pmod 7$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 10 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k - 11$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+1}{18k+9}]$
5. Give 5 students $[\frac{4k}{7} : \frac{9k+8}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-7}{7} : 1]$
6. Give 2 students $[\frac{4k+7}{7} : \frac{9k+8}{18k+9}]$ and $[5 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-21}{7} : 1]$

If $k \equiv 1 \pmod 7$ and $k \geq 2$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k - 7$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+1}{18k+9}]$
5. Give 6 students $[\frac{4k+3}{7} : \frac{9k+8}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-5}{7} : 1]$
6. Give 1 student $[\frac{4k-4}{7} : \frac{9k+8}{18k+9}]$ and $[6 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-19}{7} : 1]$ Since $k \equiv 1 \pmod 7$, $k = 1, 8, 15, \ldots$. If $k = 1$ then step 6 involves $5k - 19 = -14$ shares which is impossible.

EXCEPTION: $f(10,9) = \frac{1}{3}$. Upper bound by INT, EBM, MID.

1. 4 muffins divided $(\frac{3}{9}, \frac{3}{9}\frac{3}{9})$
2. 6 muffins divided $(\frac{4}{9}, \frac{5}{9})$
3. Give 6 students $[\frac{3}{9}, \frac{3}{9}, \frac{4}{9}]$
4. Give 3 students $[\frac{5}{9}, \frac{5}{9}]$

If $k \equiv 2 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 3 muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
3. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $5k-10$ muffins UNDIVIDED
5. Give 2 students $[2k+1 : \frac{9k+1}{18k+9}]$
6. Give 3 students $[\frac{4k+6}{7} : \frac{9k+8}{18k+9}]$ and $[1 : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-10}{7} : 1]$
7. Give 3 students $[\frac{4k-1}{7} : \frac{9k+8}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[1 : \frac{9k+7}{18k+9}]$ and $[\frac{5k-10}{7} : 1]$
8. Give 1 student $[\frac{4k-1}{7} : \frac{9k+8}{18k+9}]$ and $[3 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-10}{7} : 1]$

If $k \equiv 3 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. $5k-1$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{9k+1}{18k+9}]$
4. Give 7 students $[\frac{4k+2}{7} : \frac{9k+8}{18k+9}]$ and $[\frac{5k-1}{7} : 1]$

If $k \equiv 4 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 3 muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
3. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $5k-10$ muffins UNDIVIDED
5. Give 2 students $[2k+1 : \frac{9k+1}{18k+9}]$
6. Give 3 students $[\frac{4k-2}{7} : \frac{9k+8}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[1 : \frac{9k+7}{18k+9}]$ and $[\frac{5k-6}{7} : 1]$
7. Give 3 students $[\frac{4k+5}{7} : \frac{9k+8}{18k+9}]$ and $[1 : \frac{9k+2}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-13}{7} : 1]$
8. Give 1 student $[\frac{4k+5}{7} : \frac{9k+8}{18k+9}]$ and $[3 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-13}{7} : 1]$

If $k \equiv 5 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k-7$ muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{9k+1}{18k+9}]$
5. Give 6 students $[\frac{4k+1}{7} : \frac{9k+8}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-4}{7} : 1]$
6. Give 1 student $[\frac{4k+8}{7} : \frac{9k+8}{18k+9}]$ and $[6 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-25}{7} : 1]$

If $k \equiv 6 \pmod 7$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 10 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k - 11$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+1}{18k+9}]$
5. Give 5 students $[\frac{4k+4}{7} : \frac{9k+8}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-9}{7} : 1]$
6. Give 2 students $[\frac{4k-3}{7} : \frac{9k+8}{18k+9}]$ and $[5 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-16}{7} : 1]$

7.2 f(9k+2,9)

$f(9k+2,9) = \frac{9k+2}{18k+9}$ for $k \leq 1, 3, 4, 5$

If $k \equiv 0 \pmod 5$ and $k \neq 5$:

1. $8k+4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 4 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $k-6$ muffins UNDIVIDED
4. Give 4 students $[2k+1 : \frac{9k+2}{18k+9}]$
5. Give 4 students $[\frac{8k+5}{5} : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{k-5}{5} : 1]$
6. Give 1 student $[\frac{8k}{5} : \frac{9k+7}{18k+9}]$ and $[4 : \frac{9k+5}{18k+9}]$ and $[\frac{k-10}{5} : 1]$ (Since $k \equiv 0 \pmod 5$, $k = 5, 10, \ldots$. When $k = 5$ this step involves $k - 10 = -5$ shares which is impossible.)

EXCEPTION: $f(47, 9) = \frac{37}{78}$**. Upper Bound by INT.**

1. 40 muffins divided $(\frac{111}{234}, \frac{123}{234})$
2. 4 muffins divided $(\frac{112}{234}, \frac{122}{234})$
3. 2 muffins divided $(\frac{115}{234}, \frac{119}{234})$
3. 1 muffins divided $(\frac{117}{234}, \frac{117}{234})$
4. Give 4 students $[10 : \frac{111}{234} \ || \ 1 : \frac{112}{234}]$
5. Give 2 students $[\frac{117}{234}, \frac{122}{234}, \frac{122}{234}, \ || \ 7 : \frac{123}{234}]$
6. Give 1 students $[\frac{119}{234}, \frac{129}{234}, \ || \ 8 : \frac{123}{234}]$
7. Give 2 students $[\frac{115}{234}, \ || \ 9 : \frac{123}{234}]$

If $k \equiv 1 \pmod 5$ and $k \neq 1$:

1. $8k+4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 2 muffins divided $(\frac{9k+3}{18k+9}, \frac{9k+6}{18k+9})$
3. 2 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $k-6$ muffins UNDIVIDED
5. Give 4 students $[2k+1 : \frac{9k+2}{18k+9}]$
6. Give 2 students $[\frac{8k+7}{5} : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+3}{18k+9}]$ and $[\frac{k-6}{5} : 1]$
7. Give 2 students $[\frac{8k+2}{5} : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[1 : \frac{9k+6}{18k+9}]$ and $[\frac{k-6}{5} : 1]$
8. Give 1 student $[\frac{8k+2}{5} : \frac{9k+7}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{k-6}{5} : 1]$ (Since $k \equiv 1 \pmod 5$, $k = 1, 6, 11, \ldots$. If $k = 1$ then the steps 7 and 8 involve $k - 6 = -5$ pieces which is impossible.)

EXCEPTION: $f(11, 9) = \frac{13}{36}$. Upper Bound by INT, EBM, MID.

1. 4 muffins divided $(\frac{13}{36}, \frac{23}{36})$
2. 2 muffins divided $(\frac{14}{36}, \frac{22}{36})$
3. 4 muffins divided $(\frac{15}{36}, \frac{21}{36})$
3. 1 muffins divided $(\frac{18}{36}, \frac{18}{36})$
4. Give 2 students $[\frac{13}{36}, \frac{13}{36}, \frac{18}{36}]$
5. Give 2 students $[\frac{14}{36}, \frac{15}{36}, \frac{15}{36}]$
6. Give 1 students $[\frac{21}{36}, \frac{23}{36}]$
7. Give 4 students $[\frac{22}{36}, \frac{22}{36}]$

If $k \equiv 2 \pmod 5$:

1. $8k + 4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. $k - 2$ muffins UNDIVIDED
3. Give 4 students $[2k + 1 : \frac{9k+2}{18k+9}]$
4. Give 5 students $[\frac{8k+4}{5} : \frac{9k+7}{18k+9}]$ and $[\frac{k-2}{5} : 1]$

If $k \equiv 3 \pmod 5$ and $k \neq 3$:

1. $8k + 4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $k - 8$ muffins UNDIVIDED
4. Give 4 students $[2k + 1 : \frac{9k+2}{18k+9}]$
5. Give 3 students $[\frac{8k+6}{5} : \frac{9k+7}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[\frac{k-8}{5} : 1]$
6. Give 2 students $[\frac{8k+1}{5} : \frac{9k+7}{18k+9}]$ and $[3 : \frac{9k+5}{18k+9}]$ and $[\frac{k-8}{5} : 1]$

EXCEPTION: $f(29,9) = \frac{41}{90}$**. Upper Bound by HALF, INT, MID.**

1. 22 muffins divided $(\frac{41}{90}, \frac{49}{90})$
2. 3 muffins divided $(\frac{42}{90}, \frac{48}{90})$
3. 3 muffins divided $(\frac{44}{90}, \frac{46}{90})$
3. 1 muffins divided $(\frac{45}{90}, \frac{45}{90})$
4. Give 1 students $[\frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{44}{90}]$
5. Give 1 students $[\frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{41}{90}, \frac{42}{90}, \frac{42}{90}, \frac{42}{90}]$
6. Give 2 students $[\frac{45}{90}, \frac{49}{90}, \frac{49}{90}, \frac{49}{90}, \frac{49}{90}, \frac{49}{90}]$
7. Give 2 students $[\frac{46}{90}, \frac{48}{90}, \frac{49}{90}, \frac{49}{90}, \frac{49}{90}, \frac{49}{90}]$

If $k \equiv 4 \pmod 5$ and $k \neq 4$:

1. $8k + 4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 1 muffin divided $(\frac{9k+3}{18k+9}, \frac{9k+6}{18k+9})$
3. 3 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $k - 6$ muffins UNDIVIDED
5. Give 4 students $[2k + 1 : \frac{9k+2}{18k+9}]$
6. Give 3 students $[\frac{8k+3}{5} : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{k-4}{5} : 1]$
7. Give 1 student $[\frac{8k+3}{5} : \frac{9k+7}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[1 : \frac{9k+6}{18k+9}]$ and $[\frac{k-9}{5} : 1]$
8. Give 1 student $[\frac{8k+8}{5} : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+3}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{k-9}{5} : 1]$

(When $k = 4$ steps 7 and 8 involve -5 shares which is impossible.)

EXCEPTION: $f(38, 9) = \frac{59}{126}$**. Upper Bound by HALF, INT.**

1. 32 muffins divided $(\frac{59}{126}, \frac{67}{126})$
2. 4 muffins divided $(\frac{60}{126}, \frac{66}{126})$
3. 2 muffins divided $(\frac{63}{126}, \frac{63}{126})$
4. Give 4 students $[\frac{60}{126}, \ || \ 8 : \frac{59}{126}]$
5. Give 4 students $[\frac{63}{126}, \ || \ 7 : \frac{67}{126}]$
6. Give 1 students $[4 : \frac{66}{126} \ || \ 4 : \frac{67}{126}]$

7.3 f(9k+4,9)

$f(9k+4,9) = \frac{9k-4}{18k}$ for $k \neq 1$:

If k is odd and $k \neq 1$:

1. $2k$ muffins divided $(\frac{9k-4}{18k}, \frac{9k+4}{18k})$
2. $4k$ muffins divided $(\frac{9k-1}{18k}, \frac{9k+1}{18k})$
3. $3k+4$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k : \frac{9k+4}{18k}]$
5. 2 students get $[\frac{k+1}{2} : \frac{9k-4}{18k}]$ and $[k+2 : \frac{9k+1}{18k}]$ and $[\frac{k-3}{2} : \frac{1}{2}]$
6. 2 students get $[\frac{k-1}{2} : \frac{9k-4}{18k}]$ and $[k-2 : \frac{9k+1}{18k}]$ and $[\frac{k+7}{2} : \frac{1}{2}]$ (When $k = 1$ step 6 involves -1 shares which is impossible.)
7. 4 students get $[k : \frac{9k-1}{18k}]$ and $[k+1 : \frac{1}{2}]$

$\frac{1}{3}$-EXCEPTION: $f(13,9) = \frac{1}{3}$. Upper bound by FC, EBM.

1. 7 muffins divided $(\frac{3}{9}, \frac{3}{9}\frac{3}{9})$
2. 6 muffins divided $(\frac{4}{9}, \frac{5}{9})$
3. Give 6 students $[\frac{3}{9}, \frac{3}{9}, \frac{3}{9}, \frac{4}{9}]$
4. Give 3 students $[\frac{3}{9}, \frac{5}{9}, \frac{5}{9}]$

If k is even:

1. $2k$ muffins divided $(\frac{9k-4}{18k}, \frac{9k+4}{18k})$
2. $4k$ muffins divided $(\frac{9k-1}{18k}, \frac{9k+1}{18k})$
3. $3k+4$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k : \frac{9k+4}{18k}]$
5. 4 students get $[\frac{k}{2} : \frac{9k-4}{18k}]$ and $[k : \frac{9k+1}{18k}]$ and $[\frac{k+2}{2} : \frac{1}{2}]$
6. 4 students get $[k : \frac{9k-1}{18k}]$ and $[k+1 : \frac{1}{2}]$

7.4 f(9k+5,9)

$f(9k+5, 9) = \frac{9k+5}{18k+18}$ for $k \geq 1$:

If k is odd:

1. $2k+2$ muffins divided $(\frac{9k+5}{18k+18}, \frac{9k+13}{18k+18})$
2. $4k+4$ muffins divided $(\frac{9k+8}{18k+18}, \frac{9k+10}{18k+18})$
3. $3k-1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k+2 : \frac{9k+5}{18k+18}]$
5. 4 students get $[\frac{k+1}{2} : \frac{9k+13}{18k+18}]$ and $[k+1 : \frac{9k+8}{18k+18}]$ and $[\frac{k-1}{2} : \frac{1}{2}]$
6. 4 students get $[k+1 : \frac{9k+10}{18k+18}]$ and $[k : \frac{1}{2}]$

If k is even:

1. $2k+2$ muffins divided $(\frac{9k+5}{18k+18}, \frac{9k+13}{18k+18})$
2. 4 muffins divided $(\frac{9k+7}{18k+18}, \frac{9k+11}{18k+18})$
3. $4k-4$ muffins divided $(\frac{9k+8}{18k+18}, \frac{9k+10}{18k+18})$
4. $3k+3$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. 1 student gets $[2k+2 : \frac{9k+5}{18k+18}]$
6. 2 students get $[\frac{k+2}{2} : \frac{9k+13}{18k+18}]$ and $[2 : \frac{9k+7}{18k+18}]$ and $[k-1 : \frac{9k+8}{18k+18}]$ and $[\frac{k-2}{2} : \frac{1}{2}]$
7. 2 students get $[\frac{k}{2} : \frac{9k+13}{18k+18}]$ and $[k-1 : \frac{9k+8}{18k+18}]$ and $[\frac{k+4}{2} : \frac{1}{2}]$
8. 4 students get $[k-1 : \frac{9k+10}{18k+18}]$ and $[1 : \frac{9k+11}{18k+18}]$ and $[k+1 : \frac{1}{2}]$

7.5 f(9k+7,9)

$f(9k+7, 9) = \frac{9k+2}{18k+9}$ for $k \geq 1$:

If $k \equiv 0 \pmod 5$:

1. $8k + 4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 4 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $k - 1$ muffins UNDIVIDED
4. Give 4 students $[2k + 1 : \frac{9k+7}{18k+9}]$
5. Give 4 students $[\frac{8k+5}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{k}{5} : 1]$
6. Give 1 student $[\frac{8k}{5} : \frac{9k+2}{18k+9}]$ and $[4 : \frac{9k+4}{18k+9}]$ and $[\frac{k-5}{5} : 1]$

If $k \equiv 1 \pmod 5$:

1. $8k + 4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 2 muffins divided $(\frac{9k+3}{18k+9}, \frac{9k+6}{18k+9})$
3. 2 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $k - 1$ muffins UNDIVIDED
5. Give 4 students $[2k + 1 : \frac{9k+7}{18k+9}]$
6. Give 2 students $[\frac{8k+7}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+6}{18k+9}]$ and $[\frac{k-1}{5} : 1]$
7. Give 2 students $[\frac{8k+2}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+3}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{k-1}{5} : 1]$
8. Give 1 student $[\frac{8k+2}{5} : \frac{9k+2}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[\frac{k-1}{5} : 1]$

If $k \equiv 2 \pmod 5$:

1. $8k+4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. $k+3$ muffins UNDIVIDED
3. Give 4 students $[2k+1 : \frac{9k+7}{18k+9}]$
4. Give 5 students $[\frac{8k+4}{5} : \frac{9k+2}{18k+9}]$ and $[\frac{k+3}{5} : 1]$

If $k \equiv 3 \pmod 5$:

1. $8k+4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $k-3$ muffins UNDIVIDED
4. Give 4 students $[2k+1 : \frac{9k+7}{18k+9}]$
5. Give 3 students $[\frac{8k+6}{5} : \frac{9k+2}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{k-3}{5} : 1]$
6. Give 2 students $[\frac{8k+1}{5} : \frac{9k+2}{18k+9}]$ and $[3 : \frac{9k+4}{18k+9}]$ and $[\frac{k-3}{5} : 1]$

If $k \equiv 4 \pmod 5$:

1. $8k+4$ muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
2. 1 muffins divided $(\frac{9k+3}{18k+9}, \frac{9k+6}{18k+9})$
3. 3 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $k-1$ muffins UNDIVIDED
5. Give 4 students $[2k+1 : \frac{9k+7}{18k+9}]$
6. Give 3 students $[\frac{8k+3}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{k+1}{5} : 1]$
7. Give 1 student $[\frac{8k+3}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+3}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{k-4}{5} : 1]$
8. Give 1 student $[\frac{8k+8}{5} : \frac{9k+2}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[1 : \frac{9k+6}{18k+9}]$ and $[\frac{k-4}{5} : 1]$

7.6 f(9k+8,9)

$f(9k+8, 9) = \frac{9k+1}{18k+9}$ for $k \geq 2$:

If $k \equiv 0 \pmod 7$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 10 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k - 4$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+8}{18k+9}]$
5. Give 5 students $[\frac{4k}{7} : \frac{9k+1}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[\frac{5k}{7} : 1]$
6. Give 2 students $[\frac{4k+7}{7} : \frac{9k+1}{18k+9}]$ and $[5 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-14}{7} : 1]$

If $k \equiv 1 \pmod 7$ and $k \geq 2$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+8}{18k+9}]$
5. Give 6 students $[\frac{4k+3}{7} : \frac{9k+1}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[\frac{5k+2}{7} : 1]$
6. Give 1 student $[\frac{4k-4}{7} : \frac{9k+1}{18k+9}]$ and $[6 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-12}{7} : 1]$

If $k \equiv 2 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 3 muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
3. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $5k-3$ muffins UNDIVIDED
5. Give 2 students $[2k+1 : \frac{9k+8}{18k+9}]$
6. Give 3 students $[\frac{4k+6}{7} : \frac{9k+1}{18k+9}]$ and $[1 : \frac{9k+7}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-3}{7} : 1]$
7. Give 3 students $[\frac{4k-1}{7} : \frac{9k+1}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[1 : \frac{9k+2}{18k+9}]$ and $[\frac{5k-3}{7} : 1]$
8. Give 1 student $[\frac{4k-1}{7} : \frac{9k+1}{18k+9}]$ and $[3 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-3}{7} : 1]$

If $k \equiv 3 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. $5k+6$ muffins UNDIVIDED
3. Give 2 students $[2k+1 : \frac{9k+8}{18k+9}]$
4. Give 7 students $[\frac{4k+2}{7} : \frac{9k+1}{18k+9}]$ and $[\frac{5k+6}{7} : 1]$

If $k \equiv 4 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 3 muffins divided $(\frac{9k+2}{18k+9}, \frac{9k+7}{18k+9})$
3. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
4. $5k-3$ muffins UNDIVIDED
5. Give 2 students $[2k+1 : \frac{9k+8}{18k+9}]$
6. Give 3 students $[\frac{4k-2}{7} : \frac{9k+1}{18k+9}]$ and $[1 : \frac{9k+5}{18k+9}]$ and $[1 : \frac{9k+2}{18k+9}]$ and $[\frac{5k+1}{7} : 1]$
7. Give 3 students $[\frac{4k+5}{7} : \frac{9k+1}{18k+9}]$ and $[1 : \frac{9k+7}{18k+9}]$ and $[2 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-6}{7} : 1]$
8. Give 1 student $[\frac{4k+5}{7} : \frac{9k+1}{18k+9}]$ and $[3 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-6}{7} : 1]$

If $k \equiv 5 \pmod 7$:

1. $4k+2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 6 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k$ muffins UNDIVIDED
4. Give 2 students $[2k+1 : \frac{9k+8}{18k+9}]$
5. Give 6 students $[\frac{4k+1}{7} : \frac{9k+1}{18k+9}]$ and $[1 : \frac{9k+4}{18k+9}]$ and $[\frac{5k+3}{7} : 1]$
6. Give 1 student $[\frac{4k+8}{7} : \frac{9k+1}{18k+9}]$ and $[6 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-18}{7} : 1]$

If $k \equiv 6 \pmod 7$:

1. $4k + 2$ muffins divided $(\frac{9k+1}{18k+9}, \frac{9k+8}{18k+9})$
2. 10 muffins divided $(\frac{9k+4}{18k+9}, \frac{9k+5}{18k+9})$
3. $5k - 4$ muffins UNDIVIDED
4. Give 2 students $[2k + 1 : \frac{9k+8}{18k+9}]$
5. Give 5 students $[\frac{4k+4}{7} : \frac{9k+1}{18k+9}]$ and $[2 : \frac{9k+5}{18k+9}]$ and $[\frac{5k-2}{7} : 1]$
6. Give 2 students $[\frac{4k-3}{7} : \frac{9k+1}{18k+9}]$ and $[5 : \frac{9k+4}{18k+9}]$ and $[\frac{5k-9}{7} : 1]$

8 f(m,10)

8.1 f(10k,10)

Give every student k whole muffins.

8.2 f(10k+1,10)

$f(10k+1,10) = \frac{10k+1}{20k+10}$ for $k \geq 2$:

1. $4k+2$ muffins divided $(\frac{10k+1}{20k+10}, \frac{10k+9}{20k+10})$
2. $4k-4$ muffins divided $(\frac{10k+3}{20k+10}, \frac{10k+7}{20k+10})$
3. 4 muffins divided $(\frac{10k+4}{20k+10}, \frac{10k+6}{20k+10})$
4. $2k-1$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. Give 2 students $[2k+1 : \frac{10k+1}{20k+10}]$
6. Give 4 students $[k : \frac{10k+9}{20k+10}]$ and $[k-1 : \frac{10k+3}{20k+10}]$ and $[1 : \frac{10k+4}{20k+10}]$
7. Give 2 students $[1 : \frac{10k+9}{20k+10}]$ and $[1 : \frac{10k+6}{20k+10}]$ and $[k-2 : \frac{10k+7}{20k+10}]$ and $[k : \frac{1}{2}]$
8. Give 2 students $[1 : \frac{10k+6}{20k+10}]$ and $[k : \frac{10k+7}{20k+10}]$ and $[k-1 : \frac{1}{2}]$

EXCEPTION: $f(11,10) = \frac{7}{20}$. Upper Bound by INT, EBM,MID.

1. 4 muffins divided $(\frac{7}{20}, \frac{13}{20})$
2. 2 muffins divided $(\frac{8}{20}, \frac{12}{20})$
3. 4 muffins divided $(\frac{11}{20}, \frac{11}{20})$
4. 1 muffins divided $(\frac{10}{20}, \frac{10}{20})$
5. Give 2 students $[2 : \frac{7}{20}, \ || \ \frac{8}{20}]$
6. Give 4 students $[\frac{9}{20}, \frac{13}{20}]$
7. Give 2 students $[\frac{10}{20}, \frac{12}{20}]$
8. Give 2 students $[\frac{11}{20}, \frac{11}{20}]$

8.3 f(10k+3,10)

$f(10k+3,10) = \frac{10k-3}{20k}$ for $k \geq 2$:

8.4 f(10k+7,10)

$f(10k+7,10) = \frac{10k+7}{20k+20}$ for $k \geq 2$:

8.5 f(10k+9,10)

$f(10k+9,10) = \frac{10k+1}{20k+10}$ for $k \geq 2$:

1. $4k+2$ muffins divided $(\frac{10k+1}{20k+10}, \frac{10k+9}{20k+10})$
2. $4k-4$ muffins divided $(\frac{10k+3}{20k+10}, \frac{10k+7}{20k+10})$
3. 4 muffins divided $(\frac{10k+4}{20k+10}, \frac{10k+6}{20k+10})$
4. $2k+7$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. Give 2 students $[2k+1 : \frac{10k+9}{20k+10}]$
6. Give 4 students $[k : \frac{10k+1}{20k+10}]$ and $[k-1 : \frac{10k+7}{20k+10}]$ and $[1 : \frac{10k+6}{20k+10}]$ and $[2 : \frac{1}{2}]$
7. Give 2 students $[1 : \frac{10k+1}{20k+10}]$ and $[1 : \frac{10k+4}{20k+10}]$ and $[k-2 : \frac{10k+3}{20k+10}]$ and $[k+2 : \frac{1}{2}]$
8. Give 2 students $[1 : \frac{10k+4}{20k+10}]$ and $[k : \frac{10k+3}{20k+10}]$ and $[k+1 : \frac{1}{2}]$

9 f(m,11)

9.1 f(11k,11)

Give every student k whole muffins.

9.2 f(11k+1,11)

9.3 f(11k+2,11)

9.4 f(11k+3,11)

9.5 f(11k+4,11)

9.6 f(11k+5,11)

$f(11k+5, 11) = \frac{11k-5}{22k}$ for $k \geq 2$:

If k is odd and $k \geq 2$:

1. $2k$ muffins divided $(\frac{11k-5}{18k}, \frac{11k+5}{18k})$
2. $6k - 12$ muffins divided $(\frac{11k-1}{22k}, \frac{11k+1}{22k})$
3. 6 muffins divided $(\frac{11k-2}{22k}, \frac{11k+2}{22k})$
4. $3k + 11$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
5. 1 student gets $[2k : \frac{11k+5}{22k}]$
6. 2 students get $[\frac{k+1}{2} : \frac{11k-5}{22k}]$ and $[\frac{3k-7}{2} : \frac{11k+1}{22k}]$ and $[3 : \frac{11k+2}{22k}]$ and $[1 : \frac{1}{2}]$
7. 2 students get $[\frac{k-1}{2} : \frac{11k-5}{22k}]$ and $[\frac{3k-5}{2} : \frac{11k+1}{22k}]$ and $[4 : \frac{1}{2}]$
8. 6 students get $[1 : \frac{11k-2}{22k}]$ and $[k-2 : \frac{11k-1}{22k}]$ and $[k+2 : \frac{1}{2}]$

If k is even:

1. $2k$ muffins divided $(\frac{11k-5}{22k}, \frac{11k+5}{22k})$
2. $6k$ muffins divided $(\frac{11k-1}{22k}, \frac{11k+1}{22k})$
3. $3k + 5$ muffins divided $(\frac{1}{2}, \frac{1}{2})$
4. 1 student gets $[2k : \frac{11k+5}{22k}]$
5. 4 students get $[\frac{k}{2} : \frac{11k-5}{22k}]$ and $[\frac{3k}{2} : \frac{11k+1}{22k}]$ and $[1 : \frac{1}{2}]$
6. 6 students get $[k : \frac{11k-1}{22k}]$ and $[k+1 : \frac{1}{2}]$

9.7 f(11k+6,11)

$f(11k+6, 11) = \frac{11k+6}{22k+22}$:

If k is odd:

9.8 f(11k+7,11)

9.9 f(11k+8,11)

9.10 f(11k+9,11)

$f(11k+9, 11) = \frac{11k+2}{22k+11}$:

If $k \equiv 0 \pmod 7$:

1. $8k+4$ muffins divided $(\frac{11k+2}{22k+9}, \frac{11k+9}{22k+11})$
2. 12 muffins divided $(\frac{11k+5}{22k+11}, \frac{11k+6}{22k+11})$
3. $3k-7$ muffins UNDIVIDED
4. Give 4 students $[2k+1 : \frac{11k+9}{22k+11}]$
5. Give 4 students $[\frac{8k+7}{7} : \frac{11k+2}{22k+11}]$ and $[3 : \frac{11k+6}{22k+11}]$ and $[\frac{3k-7}{7} : 1]$
6. Give 3 students $[\frac{8k}{7} : \frac{11k+2}{22k+11}]$ and $[4 : \frac{11k+5}{22k+11}]$ and $[\frac{3k-7}{7} : 1]$

If $k \equiv 1 \pmod 7$:

If $k \equiv 2 \pmod 7$:

1. $8k+4$ muffins divided $(\frac{11k+2}{22k+11}, \frac{11k+9}{22k+11})$
2. 2 muffins divided $(\frac{11k+4}{22k+11}, \frac{11k+7}{22k+11})$
3. 4 muffins divided $(\frac{11k+5}{22k+11}, \frac{11k+6}{22k+11})$
4. $3k-1$ muffins UNDIVIDED
5. Give 4 students $[2k+1 : \frac{11k+9}{22k+11}]$
6. Give 4 students $[\frac{8k+5}{7} : \frac{11k+2}{22k+11}]$ and $[1 : \frac{11k+6}{22k+11}]$ and $[\frac{3k+1}{7} : 1]$
7. Give 2 students $[\frac{8k+5}{7} : \frac{11k+2}{22k+11}]$ and $[2 : \frac{11k+5}{22k+11}]$ and $[1 : \frac{11k+7}{22k+11}]$ and

$[\frac{3k-6}{7} : 1]$

8. Give 1 student $[\frac{8k-2}{7} : \frac{11k+2}{22k+11}]$ and $[2 : \frac{11k+4}{22k+11}]$ and $[\frac{3k+1}{7} : 1]$

If $k \equiv 3 \pmod 7$:

1. $8k + 4$ muffins divided $(\frac{11k+2}{22k+11}, \frac{11k+9}{22k+11})$
2. $3k + 5$ muffins UNDIVIDED
3. Give 4 students $[2k + 1 : \frac{11k+9}{22k+11}]$
4. Give 7 students $[\frac{8k+4}{7} : \frac{11k+2}{22k+11}]$ and $[\frac{3k+5}{7} : 1]$

If $k \equiv 4 \pmod 7$:

1. $8k + 4$ muffins divided $(\frac{11k+2}{22k+9}, \frac{11k+9}{22k+11})$
2. 2 muffins divided $(\frac{11k+4}{22k+11}, \frac{11k+7}{22k+11})$
3. 4 muffins divided $(\frac{11k+5}{22k+11}, \frac{11k+6}{22k+11})$
4. $3k - 1$ muffins UNDIVIDED
5. Give 4 students $[2k + 1 : \frac{11k+9}{22k+11}]$
6. Give 4 students $[\frac{8k+3}{7} : \frac{11k+2}{22k+11}]$ and $[1 : \frac{11k+5}{22k+11}]$ and $[\frac{3k+2}{7} : 1]$
7. Give 2 students $[\frac{8k+3}{7} : \frac{11k+2}{22k+11}]$ and $[1 : \frac{11k+4}{22k+11}]$ and $[2 : \frac{11k+6}{22k+11}]$ and $[\frac{3k-5}{7} : 1]$
8. Give 1 student $[\frac{8k+10}{7} : \frac{11k+2}{22k+11}]$ and $[2 : \frac{11k+7}{22k+11}]$ and $[\frac{3k-5}{7} : 1]$

If $k \equiv 5 \pmod 7$:

1. $8k + 4$ muffins divided $(\frac{11k+2}{22k+9}, \frac{11k+9}{22k+11})$
2. 2 muffins divided $(\frac{11k+3}{22k+11}, \frac{11k+8}{22k+11})$
3. 6 muffins divided $(\frac{11k+5}{22k+11}, \frac{11k+6}{22k+11})$
4. $3k - 3$ muffins UNDIVIDED
5. Give 4 students $[2k + 1 : \frac{11k+9}{22k+11}]$
6. Give 3 students $[\frac{8k+2}{7} : \frac{11k+2}{22k+11}]$ and $[2 : \frac{11k+5}{22k+11}]$ and $[\frac{3k-1}{7} : 1]$

7. Give 2 students $[\frac{8k+2}{7} : \frac{11k+2}{22k+11}]$ and $[1 : \frac{11k+3}{22k+11}]$ and $[3 : \frac{11k+6}{22k+11}]$ and $[\frac{3k-8}{7} : 1]$

8. Give 2 student $[\frac{8k+9}{7} : \frac{11k+2}{22k+11}]$ and $[1 : \frac{11k+8}{22k+11}]$ and $[\frac{3k-1}{7} : 1]$

If $k \equiv 6 \pmod 7$:

1. $8k + 4$ muffins divided $(\frac{11k+2}{22k+9}, \frac{11k+9}{22k+11})$
2. 12 muffins divided $(\frac{11k+5}{22k+11}, \frac{11k+6}{22k+11})$
3. $3k - 7$ muffins UNDIVIDED
4. Give 4 students $[2k + 1 : \frac{11k+9}{22k+11}]$
5. Give 4 students $[\frac{8k-6}{7} : \frac{11k+2}{22k+11}]$ and $[3 : \frac{11k+5}{22k+11}]$ and $[\frac{3k-4}{7} : 1]$
6. Give 3 students $[\frac{8k+1}{7} : \frac{11k+2}{22k+11}]$ and $[4 : \frac{11k+6}{22k+11}]$ and $[\frac{3k-11}{7} : 1]$