The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:
The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
## 5 Muffins, 3 Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
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<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
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</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
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</tbody>
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**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?  
Work on it with your neighbor
### 5 Muffins, 3 People–Proc by Picture

**YES WE CAN!**

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

*Is there a procedure with a larger smallest piece?*

*Work on it with your neighbor*
NO WE CAN’T!

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

**Case 0:** Some muffin is uncut. Cut it \( (\frac{1}{2}, \frac{1}{2}) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases. (**Henceforth:** All muffins cut into \( \geq 2 \) pieces.)
5 Muffins, 3 People–Can’t Do Better Than $\frac{5}{12}$

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Case 1: Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (*Henceforth:* All muffins cut into 2 pieces.)
5 Muffins, 3 People–Can’t Do Better Than \( \frac{5}{12} \)

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There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

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Case 1: Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \). (Henceforth: All muffins cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets \( \geq 4 \) pieces. He has some piece

\[
\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}
\]

Great to see \( \frac{5}{12} \)
General Problem

\( f(m, s) \) be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide \( m \) muffins among \( s \) students so that everyone gets \( \frac{m}{s} \).

We have shown \( f(5, 3) = \frac{5}{12} \) here.

We have shown \( f(m, s) \) exists, is rational, and is computable using a Mixed Int Program (in paper).
1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant!

Have **General Theorems** from which **upper bounds** follow. Have **General Procedures** from which **lower bounds** follow.
Clearly $f(3, 5) \geq \frac{1}{5}$.

Can we get $f(3, 5) > \frac{1}{5}$?

Work on it with your neighbor.
\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

Can we do better? Work on it with your neighbor.
\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
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3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 student \((\frac{6}{20}, \frac{6}{20})\)

**Can we do better?**

**Work on it with your neighbor**
NO WE CAN’T!

There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$. 
NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.

(*Henceforth:* All people get $\geq 2$ pieces.)
NO WE CAN’T!
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( **Henceforth:** All people get $\geq 2$ pieces.)

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(\textbf{Henceforth:} All people get $\geq 2$ pieces.)

Case 1: Alice gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.
(\textbf{Henceforth:} Everyone gets 2 pieces.)

Case 2: Everyone gets 2 pieces. 10 pieces, 3 muffins:
\textbf{Some muffin} gets $\geq 4$ pieces. So some piece is $\leq \frac{1}{4}$. 

3 Muffins, 5 People–Can’t Do Better Than $\frac{1}{4}$
\( f(3, 5) \) and \( f(5, 3) \)

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

Duality Theorem:
\[ f(\text{m}, \text{s}) = \frac{\text{m}}{\text{s}} f(\text{s}, \text{m}) \]
**f(3, 5) and f(5, 3)**

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
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4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)

**Duality Theorem:**

\[ f(m, s) = m \cdot s \cdot f(s, m) \]
**f(3, 5) and f(5, 3)**

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
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\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
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Duality Theorem:

\[ f(m, s) = m \cdot s \]

\[ f(s, m) \]
\( f(3, 5) \) and \( f(5, 3) \)

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
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4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
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\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by \( \frac{3}{5} \).
\[ f(3, 5) \] and \[ f(5, 3) \]

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([5/12, 7/12]\)
2. Divide 1 muffin \([6/12, 6/12]\)
3. Give 2 students \((6/12, 7/12, 7/12)\)
4. Give 1 student \((5/12, 5/12, 5/12, 5/12)\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffins \([6/20, 7/20, 7/20]\)
2. Divide 1 muffin \([5/20, 5/20, 5/20, 5/20]\)
3. Give 4 students \((5/20, 7/20)\)
4. Give 1 student \((6/20, 6/20)\)

\(f(3, 5)\) proc is \(f(5, 3)\) proc but swap Divide/Give and mult by \(3/5\).

**Duality Theorem:** \[ f(m, s) = \frac{m}{s} f(s, m). \]
We know and use the following:

1. By duality can assume $m > s$
2. If $s$ divides $m$ then $f(m, s) = 1$ so assume $s$ does not divide $m$.
3. All muffins are cut in $\geq 2$ pcs. Replace uncut muff with $2 \frac{1}{2}$’s
4. If assuming $f(m, s) > \alpha > \frac{1}{3}$, assume all muffin in $\leq 2$ pcs.
5. $f(m, s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

We do not know this but still use: $f(m, s)$ only depends on $\frac{m}{s}$.

All of our techniques that hold for $(m, s)$ hold for $(Am, As)$.
For particular numbers, we only look at $m, s$ rel prime.
FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

\[ f(m, s) \leq FC(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lfloor 2m/s \rfloor}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}. \]

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. \( \exists \) piece $\leq \frac{m}{s} \times \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

**Someone** gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. \( \exists \) piece $\geq \frac{m}{s \lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

Note: A Mod 3 Pattern.

Theorem: For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Note: A Mod 4 Pattern.

**Theorem:** For all \( m \geq 4 \), \( f(m, 4) = \text{FC}(m, 4) \).

**FC-Conjecture:** For all \( m, s \) with \( m \geq s \), \( f(m, s) = \text{FC}(m, s) \).
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = \text{FC}(m, 5)$. 
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \left\lceil \frac{22}{5} \right\rceil}, 1 - \frac{11}{5 \left\lfloor \frac{22}{5} \right\rfloor} \right\} \right\} = \frac{11}{25}. \]
What About FIVE students, ELEVEN muffins?

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f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.
\]

We tried to find a protocol to divide 11 muffins for 5 people, each gets \( \frac{11}{5} \), and smallest piece is size \( \frac{11}{25} = 0.44 \).
What About FIVE students, ELEVEN muffins?

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We tried to find a protocol to divide 11 muffins for 5 people, each gets \(\frac{11}{5}\), and smallest piece is size \(\frac{11}{25} = 0.44\).

We found a protocol with smallest piece \(\frac{13}{30} = 0.4333\ldots\):

1. Divide 1 muffin \((\frac{15}{30}, \frac{15}{30})\).
2. Divide 2 muffins \((\frac{14}{30}, \frac{16}{30})\).
3. Divide 8 muffins \((\frac{13}{30}, \frac{17}{30})\).
4. Give 2 students \([\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]\).
5. Give 1 students \([\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]\).
6. Give 2 students \([\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]\).
We have:

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots \]
So Now What?

We have:

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff= 0.006666 \ldots} \]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

Vote
So Now What?

We have:

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots
\]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
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Vote \( \text{WE SHOW: } f(11, 5) = \frac{13}{30} \). \textbf{Exciting} new technique!
Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$. The other piece from muffin $M$ is the *buddy of $x$*. 

Note that the buddy of $x$ is of size

$$1 - x.$$
There is a procedure for 11 muffins, 5 students where each student gets \(\frac{11}{5}\) muffins, smallest piece \(N\). We want \(N \leq \frac{13}{30}\).

**Case 0:** Some muffin is uncut. Cut it \((\frac{1}{2}, \frac{1}{2})\) and give both halves to whoever got the uncut muffin. Reduces to other cases.
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( (\frac{1}{2}, \frac{1}{2}) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

(*Negation of Case 0 and Case 1: All muffins cut into 2 pieces.*)
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$ 

**Case 3:** Some student gets $\leq 3$ pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$ 

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$ 

(*Negation of Cases 2 and 3:* Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a 4-share.
We call a share that goes to a person who gets 5 shares a 5-share.
Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w, x, y, z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let $x$ be the largest of $x, y, z$

$$x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at buddy of $x$.

$$B(x) \leq 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!
Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are $\geq 12$ pieces $> \frac{1}{2}$. Can’t occur.
Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the INT method. We give a more sophisticated example
More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$. 

Case 1: A student gets $\geq 6$ shares. Some piece $\leq 24 \times 6 < 19\frac{11}{44}$.

Case 2: A student gets $\leq 3$ shares. Some piece $\geq 24 \times 3 = 8\frac{3}{11}$. Buddy of that piece $\leq 1 - 8\frac{3}{11} \leq 3\frac{1}{11} < 19\frac{11}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
Assume (24, 11)-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.
More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

**Case 2:** A student gets $\leq 3$ shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$. Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$. 
Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

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**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. \( s_4 \) is the number of them.
5-students: a student who gets 5 shares. \( s_5 \) is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

\[
4s_4 + 5s_5 = 48
\]
\[
s_4 + s_5 = 11
\]

\( s_4 = 7 \). Hence there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.
\( s_5 = 4 \). Hence there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1:
There is a share $\geq \frac{25}{44}$. Then its buddy is $\leq 1 - \frac{25}{44} = \frac{19}{44}$.

Case 3.2:
There is a share $\leq \frac{19}{44}$. Duh.

Henceforth assume that all shares are in $\left(\frac{19}{44}, \frac{25}{44}\right)$. 
Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share \( \geq \frac{25}{44} \). Then its buddy is

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]

Case 3.2: There is a share \( \leq \frac{19}{44} \). Duh.

Henceforth assume that all shares are in

\[
\left( \begin{array}{c} \frac{19}{44} \\ \frac{25}{44} \end{array} \right)
\]

\[
\left( \begin{array}{cc} \frac{19}{44} & \frac{25}{44} \\ \frac{19}{44} & \frac{25}{44} \end{array} \right)
\]
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.
Since $A + B + C + D + E = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$A + B + C + D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$. 

$$\left(\frac{19}{44}, \frac{20}{44}\right)[?? \hspace{1em} 5\text{-shs}]$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.

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$$A + B + C + D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

$$\left(\frac{19}{44}, ?? \ 5\text{-shs} \right)[\frac{20}{44} \frac{25}{44}]$$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$. Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44}\right)$$.
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \( \left( \frac{21}{44}, \frac{25}{44} \right) \), 5-shares in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).

\[
\begin{pmatrix}
\frac{19}{44} & \text{?? 5-shs} & 0 \text{ shs} & \frac{21}{44} & \text{?? 4-shs} \\
\frac{20}{44} & & & \frac{25}{44}
\end{pmatrix}
\]
Case 3.5: 4-shares in \( \left( \frac{21}{44}, \frac{25}{44} \right) \), 5-shares in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).

\[
\begin{pmatrix}
19/44 & 20/44 \\
\text{?? 5-shs} & 0 \text{ shs} & \text{?? 4-shs} & 21/44 & 25/44
\end{pmatrix}
\]

Recall: there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.
Recall: there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.
Case 3.5: 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\begin{pmatrix}
?? 5\text{-shs} \\
\frac{19}{44} \\
\end{pmatrix}\begin{pmatrix}
0 \text{ shs} \\
\frac{20}{44} \\
\end{pmatrix}\begin{pmatrix}
?? 4\text{-shs} \\
\frac{21}{44} \\
\end{pmatrix}\begin{pmatrix}
\frac{25}{44}
\end{pmatrix}
\]

Recall: there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

Recall: there are \(5s_5 = 5 \times 4 = 20\) 5-shares.

\[
\begin{pmatrix}
20 5\text{-shs} \\
\frac{19}{44} \\
\end{pmatrix}\begin{pmatrix}
0 \text{ shs} \\
\frac{20}{44} \\
\end{pmatrix}\begin{pmatrix}
28 4\text{-shs} \\
\frac{21}{44} \\
\end{pmatrix}\begin{pmatrix}
\frac{25}{44}
\end{pmatrix}
\]
More Refined Picture of What is Going On

\[
\begin{align*}
( & \frac{20}{44} \text{ 5-shs} & )[ & 0 \text{ shs} & ]( & \frac{28}{44} \text{ 4-shs} & ) \\
& \frac{19}{44} & & \frac{20}{44} & & \frac{21}{44} & & \frac{25}{44}
\end{align*}
\]

**Claim 1:** There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).
More Refined Picture of What is Going On

\[
\begin{pmatrix}
20 & 5\text{-shs} \\
19 & 44
\end{pmatrix}
\begin{pmatrix}
0 & \text{shs} \\
20 & 44
\end{pmatrix}
\begin{pmatrix}
28 & 4\text{-shs} \\
21 & 44
\end{pmatrix}
\begin{pmatrix}
25 & 44
\end{pmatrix}
\]

Claim 1: There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).

The following picture captures what we know so far.

\[
\begin{pmatrix}
20 & 5\text{-shs} \\
19 & 44
\end{pmatrix}
\begin{pmatrix}
0 \\
20 & 44
\end{pmatrix}
\begin{pmatrix}
8 & \text{S4-shs} \\
21 & 44
\end{pmatrix}
\begin{pmatrix}
0 \\
23 & 44
\end{pmatrix}
\begin{pmatrix}
20 & \text{L4-shs} \\
24 & 44
\end{pmatrix}
\begin{pmatrix}
25 & 44
\end{pmatrix}
\]

S4 = Small 4-shares
L4 = Large 4-shares. L4 shares, 5-share: buddies, so \(|L4| = 20\).
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.
\]
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.
\]

Contradiction: Each 4-student gets \( \geq 3 \) L4 shares. There are

\( s_4 = 7 \) 4-students. Hence there are \( \geq 21 \) L4-shares. But there are only 20.
INT Technique

INT is generalization of \( f(24, 11) \leq \frac{19}{44} \) proof.

**Definition:** Let \( \text{INT}(m, s) \) be the bound obtained.

1. INT proofs can get more complicated than this one.
2. \( \text{INT}(m, s) \) can be computed in \( O\left(\frac{2^m \log m}{s}\right) \). Note: do not need to know the answer ahead of time.
3. For \( 1 \leq s \leq 60, \ s < m \leq 70, \ m, s \text{ rel prime:} \)
   3.1 There are 1360 cases total.
   3.2 For 927 of the \((m, s)\), \( f(m, s) = \text{FC}(m, s) \). \( \sim 68\% \)
   3.3 For 268 of the \((m, s)\), \( f(m, s) = \text{INT}(m, s) \). \( \sim 20\% \)
   3.4 The cases not covered use **interesting** new techniques!
Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$.
Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$.
By INT-technique methods obtain:
$s_3 = 14, s_4 = 5.$

\[
\begin{pmatrix}
\frac{54}{133} & 0 & \frac{22}{133} & 0 & \frac{20}{133} \\
\frac{55}{133} & \frac{59}{133} & \frac{74}{133} & \frac{78}{133} & \frac{79}{133}
\end{pmatrix}
\]

We just look at the 3-shares:

\[
\begin{pmatrix}
\frac{59}{133} & 0 & \frac{22}{133} & 0 & \frac{20}{133} \\
\frac{74}{133} & \frac{78}{133} & \frac{79}{133}
\end{pmatrix}
\]
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\[
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 \\
\frac{74}{133} & 20 \text{ L3-shs} & \frac{79}{133}
\end{pmatrix}
\]

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) (\( |J_1| = |J_2| \))
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) (\( |J_3| = 20 \))

**Note:** Split the shares of size 66.5 between \( J_1 \) and \( J_2 \).

**Notation:** An \( e(1, 1, 3) \) students is a student who has a \( J_1 \)-share, a \( J_1 \)-share, and a \( J_3 \)-share. Generalize to \( e(i, j, k) \) easily.
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) (\( |J_1| = |J_2| \))
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) (\( |J_3| = 20 \))

1) Only students allowed: \( e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3) \). All others have either \(< \frac{31}{19}\) or \(> \frac{31}{19}\).

2) No shares in \([\frac{61}{133}, \frac{64}{133}]\). Look at \( J_1 \)-shares:
   An \( e(1, 2, 3) \)-student has \( J_1 \)-share \(> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133} \).
   An \( e(1, 3, 3) \)-student has \( J_1 \)-share \(< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133} \).

3) No shares in \([\frac{69}{133}, \frac{72}{133}]\): \( x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}] \).
GAPS Technique:  $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{61}{133})$
2. $J_2 = (\frac{64}{133}, \frac{66.5}{133})$
3. $J_3 = (\frac{66.5}{133}, \frac{69}{133})$ ($|J_2| = |J_3|$)
4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ ($|J_1| = |J_4|$)
5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ ($|J_5| = 20$)

The following are the only students who are allowed.
e(1, 5, 5).
e(2, 4, 5).
e(3, 4, 5).
e(4, 4, 4).
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be $x$
$e(2, 4, 5)$. Let the number of such students be $y_1$
$e(3, 4, 5)$. Let the number of such students be $y_2$.
$e(4, 4, 4)$. Let the number of such students be $z$.

1) $|J_2| = |J_3|$, only students using $J_2$ are $e(2, 4, 5)$ – they use one share each, only students using $J_3$ are $e(3, 4, 5)$ – they use one share each. Hence $y_1 = y_2$. We call them both $y$.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$. Contradiction.
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**—pieces add to $1$: $\{ \frac{6}{12}, \frac{6}{12} \}, \{ \frac{5}{12}, \frac{7}{12} \}$. Vectors $\{ \frac{6}{12}, \frac{6}{12} \}$ is $(0, 2, 0)$, $m_1$ muffins of this type. $\{ \frac{5}{12}, \frac{7}{12} \}$ is $(1, 0, 1)$, $m_2$ muffins of this type.

3) **Student**—pieces add to $\frac{5}{3}$ $\{ \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \}$ is $(0, 1, 2)$, $s_1$ students of this type. $\{ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \}$ is $(4, 0, 0)$, $s_2$ students of this type.

4) **Set up equations:**

$m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)$
$m_1 + m_2 = 5$
$s_1 + s_2 = 3$

**Natural Number Solution:** $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$
Matrix Technique

Want proc for \( f(m, s) \geq \frac{a}{b} \).

1) **Guess** that the only piece sizes are \( \frac{a}{b}, \ldots, \frac{b-a}{b} \)

2) **Muffin**=pieces add to 1: Vectors \( \vec{v}_i \). \( x \) types.
   \( m_i \) muffins of type \( \vec{v}_i \)

3) **Student**=pieces add to \( \frac{m}{s} \): Vectors \( \vec{u}_j \). \( y \) types.
   \( s_j \) students of type \( \vec{u}_j \)

4) **Set up equations:**
   \[
   m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x = s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y
   \]
   \[
   m_1 + \cdots + m_x = m
   \]
   \[
   s_1 + \cdots + s_y = s
   \]

5) **Look for Nat Numb sol.** If find can translate into procedure.
1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input \( m, s \), found \( f(m, s) \) and the procedure REALLY FAST.

2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob’s code is also REALLY FAST.

3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott’s algorithm was fast (poly in \( m, s \)).

4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arXiv.

5. One corollary of the work: \( f(m, s) \) only depends on \( m/s \).
Accomplishment I Am Most Proud of:

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Convinced

4 High School students (Guang, Naveen, Naveen, Sunny)

3 college student (Erik, Jacob, Daniel)

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that the most important field of Mathematics is Muffinry.
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