

The Muffin Problem

William Gasarch - University of MD

Erik Metz - University of MD

Jacob Prinz-University of MD

Daniel Smolyak- University of MD

How it Began

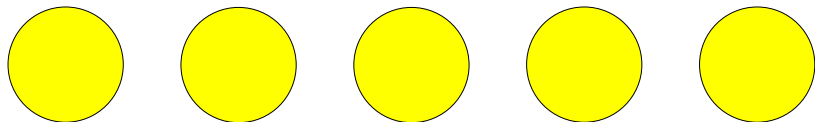
A Recreational Math Conference (Gathering for Gardner) May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

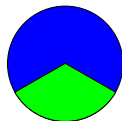
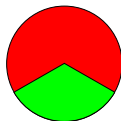
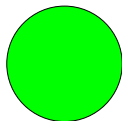
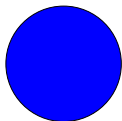
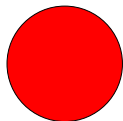
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



5 Muffins, 3 Students, Proc by Picture

Person	Color	What they Get
Alice	RED	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	BLUE	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

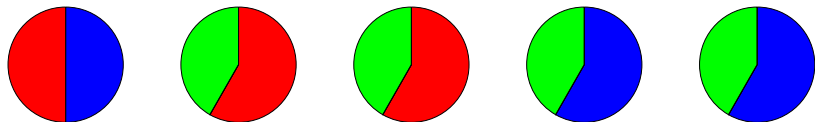
Work on it with your neighbor

5 Muffins, 3 People—Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	RED	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	BLUE	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

5 Muffins, 3 People—Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into ≥ 2 pieces.)

5 Muffins, 3 People—Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

5 Muffins, 3 People—Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \quad \text{Great to see } \frac{5}{12}$$

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

This seemed like a nice exercise and it was.

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

This seemed like a nice exercise and it was.

There can't be much more to this.

If there is not much more to this then how come

`https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170`

If there is not much more to this then how come

`https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170`

The following happened:

If there is not much more to this then how come

`https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170`

The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).

If there is not much more to this then how come

`https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170`

The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
- ▶ Come across a problem where the techniques do not work.

If there is not much more to this then how come

`https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170`

The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
- ▶ Come across a problem where the techniques do not work.
- ▶ Find a new technique **which was interesting**.

If there is not much more to this then how come

<https://www.amazon.com/Mathematical-Muffin-Morsels-Problem-Mathematics/dp/9811215170>

The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
- ▶ Come across a problem where the techniques do not work.
- ▶ Find a new technique **which was interesting**.
- ▶ Lather, Rinse, Repeat.

General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program.

General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program.

This was a case of a Theorem in **Applied Math** being used to prove a Theorem in **Pure Math**.

Amazing Results! / Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

Amazing Results! / Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer

Amazing Results! / Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer
by Co-author Erik Metz is a muffin savant !

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer
by Co-author Erik Metz is a **muffin savant** !

Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

1. By Duality Theorem can assume $m > s$

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s
4. If assuming $f(m, s) > \alpha > \frac{1}{3}$, assume all muffin in ≤ 2 pcs.

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s
4. If assuming $f(m, s) > \alpha > \frac{1}{3}$, assume all muffin in ≤ 2 pcs.
5. $f(m, s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor}\right\}\right\}.$$

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. \exists piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. \exists piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

Someone gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. \exists piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. \exists piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

Someone gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. \exists piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

$$f(3k, 3) = 1.$$

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

$$f(3k, 3) = 1.$$

$$f(3k + 1, 3) = \frac{3k-1}{6k}, k \geq 1.$$

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

$$f(3k, 3) = 1.$$

$$f(3k + 1, 3) = \frac{3k-1}{6k}, k \geq 1.$$

$$f(3k + 2, 3) = \frac{3k+2}{6k+6}.$$

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

$$f(3k, 3) = 1.$$

$$f(3k + 1, 3) = \frac{3k-1}{6k}, k \geq 1.$$

$$f(3k + 2, 3) = \frac{3k+2}{6k+6}.$$

Note: A Mod 3 Pattern.

Theorem: For all $m \geq 3$, $f(m, 3) = \text{FC}(m, 3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

$$f(4k + 2, 4) = \frac{1}{2}.$$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

$$f(4k + 2, 4) = \frac{1}{2}.$$

$$f(4k + 3, 4) = \frac{4k+1}{8k+4}.$$

Note: A Mod 4 Pattern.

Theorem: For all $m \geq 4$, $f(m, 4) = \text{FC}(m, 4)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

$$f(4k + 2, 4) = \frac{1}{2}.$$

$$f(4k + 3, 4) = \frac{4k+1}{8k+4}.$$

Note: A Mod 4 Pattern.

Theorem: For all $m \geq 4$, $f(m, 4) = \text{FC}(m, 4)$.

FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = \text{FC}(m, s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ **except $m=11$** , $f(m, 5) = \text{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

$$f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.$$

What About FIVE students, ELEVEN muffins?

$$f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.$$

We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

What About FIVE students, ELEVEN muffins?

$$f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lfloor 22/5 \rfloor}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.$$

We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

We found a protocol with smallest piece $\frac{13}{30} = 0.4333\dots$

1. Divide 1 muffin $(\frac{15}{30}, \frac{15}{30})$.
2. Divide 2 muffins $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 8 muffins $(\frac{13}{30}, \frac{17}{30})$.
4. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]$
5. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

Vote

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

Vote WE SHOW: $f(11, 5) = \frac{13}{30}$. Exciting new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M .

The *other piece* from muffin M is the *buddy of x* .

Note that the buddy of x is of size

$$1 - x.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N . We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N . We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.

We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z .

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z .

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$$f(11, 5) = \frac{13}{30}, \text{ Fun Cases}$$

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares.
There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $m = 3, 4, 5, \dots, 23$.

INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $m = 3, 4, 5, \dots, 23$.

Then we found a case where neither FC nor HALF worked.

INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $m = 3, 4, 5, \dots, 23$.

Then we found a case where neither FC nor HALF worked.

We found a new method: INT.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$ -procedure with smallest piece $> \frac{19}{44}$.

Can assume all muffin cut in two and all student gets ≥ 2 shares.

We show that there is a piece $\leq \frac{19}{44}$.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$ -procedure with smallest piece $> \frac{19}{44}$.

Can assume all muffin cut in two and all student gets ≥ 2 shares.

We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$ -procedure with smallest piece $> \frac{19}{44}$.

Can assume all muffin cut in two and all student gets ≥ 2 shares.

We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$.

Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$ -procedure with smallest piece $> \frac{19}{44}$.

Can assume all muffin cut in two and all student gets ≥ 2 shares.

We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$.

Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.

5-students: a student who gets 5 shares. s_5 is the number of them.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.

5-students: a student who gets 5 shares. s_5 is the number of them.

4-share: a share that a 4-student who gets.

5-share: a share that a 5-student who gets.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.

5-students: a student who gets 5 shares. s_5 is the number of them.

4-share: a share that a 4-student who gets.

5-share: a share that a 5-student who gets.

$$4s_4 + 5s_5 = 48$$

$$s_4 + s_5 = 11$$

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.

5-students: a student who gets 5 shares. s_5 is the number of them.

4-share: a share that a 4-student who gets.

5-share: a share that a 5-student who gets.

$$4s_4 + 5s_5 = 48$$

$$s_4 + s_5 = 11$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.

$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.

Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44} \right)$$

$$\left(\frac{19}{44}, \frac{25}{44} \right)$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-share is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $\leq v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-share is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-share is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-share is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $\leq v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-share is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $\leq v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $\leq v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

$$\left(\begin{array}{c} \frac{19}{44} \\ \text{?? 5-shs} \\ \frac{20}{44} \end{array} \right) \left[\begin{array}{c} \\ \\ \frac{25}{44} \end{array} \right)$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $\leq v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Since $v + w + x + y + z = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

$$\left(\begin{array}{c} \frac{19}{44} \\ \text{?? 5-shs} \\ \frac{20}{44} \end{array} \right) \left[\begin{array}{c} \\ \\ \frac{25}{44} \end{array} \right)$$

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-share is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-share is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of z is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44} \right).$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\left(\begin{array}{c} ?? \\ \frac{19}{44} \end{array} \right) \begin{array}{c} 5\text{-shs} \\] \end{array} \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} ?? \\ \frac{21}{44} \end{array} \right) \begin{array}{c} 4\text{-shs} \\) \\ \frac{25}{44} \end{array}$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\left(\begin{array}{c} ?? \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} ?? \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\left(\begin{array}{c} ?? \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} ?? \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

More Refined Picture of What is Going On

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

More Refined Picture of What is Going On

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

Claim 1: There are no shares $x \in \left[\frac{23}{44}, \frac{24}{44} \right]$.

More Refined Picture of What is Going On

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right]$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.

More Refined Picture of What is Going On

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.

The following picture captures what we know so far.

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

More Refined Picture of What is Going On

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.

The following picture captures what we know so far.

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

S4= Small 4-shares

L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Claim 2: Every 4-student has at least 3 L4 shares.

Diagram

$$\binom{20 \text{ 5-shs}}{\frac{19}{44}} \binom{0}{\frac{20}{44}} \binom{8 \text{ S4-shs}}{\frac{21}{44}} \binom{0}{\frac{23}{44}} \binom{20 \text{ L4-shs}}{\frac{24}{44}} \binom{0}{\frac{25}{44}}$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \binom{23}{44} + 2 \times \binom{25}{44} = \frac{24}{11}.$$

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44} \right) + 2 \times \left(\frac{25}{44} \right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets ≥ 3 L4 shares.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{23}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{21}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44} \right) + 2 \times \left(\frac{25}{44} \right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets ≥ 3 L4 shares.

There are $s_4 = 7$ 4-students.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44} \right) + 2 \times \left(\frac{25}{44} \right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets ≥ 3 L4 shares.

There are $s_4 = 7$ 4-students.

Hence there are ≥ 21 L4-shares.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44} \right) + 2 \times \left(\frac{25}{44} \right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets ≥ 3 L4 shares.

There are $s_4 = 7$ 4-students.

Hence there are ≥ 21 L4-shares. But there are only 20.

GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.

GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.

FC or HALF or INT worked on everything with $m = 3, 4, 5, \dots, 30$.

GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.

FC or HALF or INT worked on everything with $m = 3, 4, 5, \dots, 30$.

Then we found a case where neither FC nor HALF nor INT worked.

GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.

FC or HALF or INT worked on everything with $m = 3, 4, 5, \dots, 30$.

Then we found a case where neither FC nor HALF nor INT worked.

We found a new method: GAP.

Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$.

Assume $(31, 19)$ -procedure with smallest piece $> \frac{54}{133}$.

Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$.

Assume (31, 19)-procedure with smallest piece $> \frac{54}{133}$.

By INT-technique methods obtain:

$$s_3 = 14, s_4 = 5.$$

$$\left(\begin{array}{c} 20 \text{ 4-shs} \\ \frac{54}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{133} \end{array} \right] \left(\begin{array}{c} 22 \text{ S3 shs} \\ \frac{59}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{74}{133} \end{array} \right] \left(\begin{array}{c} 20 \text{ L3-shs} \\ \frac{78}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{79}{133} \end{array} \right]$$

We just look at the 3-shares:

Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$.

Assume (31, 19)-procedure with smallest piece $> \frac{54}{133}$.

By INT-technique methods obtain:

$$s_3 = 14, s_4 = 5.$$

$$\left(\begin{array}{c} 20 \text{ 4-shs} \\ \frac{54}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{133} \end{array} \right] \left(\begin{array}{c} 22 \text{ S3 shs} \\ \frac{59}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{74}{133} \end{array} \right] \left(\begin{array}{c} 20 \text{ L3-shs} \\ \frac{78}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{79}{133} \end{array} \right]$$

We just look at the 3-shares:

$$\left(\begin{array}{c} 22 \text{ S3 shs} \\ \frac{59}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{74}{133} \end{array} \right] \left(\begin{array}{c} 20 \text{ L3-shs} \\ \frac{78}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{79}{133} \end{array} \right]$$

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$$\left(\frac{59}{133} \quad 22 \text{ S3-shs} \right) \left[\frac{74}{133} \quad 0 \right] \left(\frac{78}{133} \quad 20 \text{ L3-shs} \right) \left[\frac{79}{133} \right]$$

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$$\left(\begin{array}{c} 22 \text{ S3 shs} \\ \frac{59}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{74}{133} \end{array} \right] \left(\begin{array}{c} 20 \text{ L3-shs} \\ \frac{78}{133} \end{array} \right) \left(\begin{array}{c} \\ \frac{79}{133} \end{array} \right)$$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133} \right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133} \right)$ ($|J_3| = 20$)

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$$\left(\begin{array}{c} 22 \text{ S3 shs} \\ \frac{59}{133} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{74}{133} \end{array} \right] \left(\begin{array}{c} 20 \text{ L3-shs} \\ \frac{79}{133} \end{array} \right)$$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133} \right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133} \right)$ ($|J_3| = 20$)

Notation: An $e(1, 1, 3)$ student is a student who has
a J_1 -share, a J_1 -share, and a J_3 -share.

Generalize to $e(i, j, k)$ easily.

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133}\right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_3| = 20$)

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133}\right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_3| = 20$)

1) Only students allowed: $e(1, 2, 3)$, $e(1, 3, 3)$, $e(2, 2, 2)$, $e(2, 2, 3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133}\right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_3| = 20$)

1) Only students allowed: $e(1, 2, 3)$, $e(1, 3, 3)$, $e(2, 2, 2)$, $e(2, 2, 3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $\left[\frac{61}{133}, \frac{64}{133}\right]$. Look at J_1 -shares:

An $e(1, 2, 3)$ -student has J_1 -share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.

An $e(1, 3, 3)$ -student has J_1 -share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{66.5}{133}\right)$
2. $J_2 = \left(\frac{66.5}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_3| = 20$)

1) Only students allowed: $e(1, 2, 3)$, $e(1, 3, 3)$, $e(2, 2, 2)$, $e(2, 2, 3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $\left[\frac{61}{133}, \frac{64}{133}\right]$. Look at J_1 -shares:

An $e(1, 2, 3)$ -student has J_1 -share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.

An $e(1, 3, 3)$ -student has J_1 -share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

3) No shares in $\left[\frac{69}{133}, \frac{72}{133}\right]$: $x \in \left[\frac{69}{133}, \frac{72}{133}\right] \implies 1 - x \in \left[\frac{61}{133}, \frac{64}{133}\right]$.

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{61}{133}\right)$
2. $J_2 = \left(\frac{64}{133}, \frac{66.5}{133}\right)$
3. $J_3 = \left(\frac{66.5}{133}, \frac{69}{133}\right)$ ($|J_2| = |J_3|$)
4. $J_4 = \left(\frac{72}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_4|$)
5. $J_5 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_5| = 20$)

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left(\frac{59}{133}, \frac{61}{133}\right)$
2. $J_2 = \left(\frac{64}{133}, \frac{66.5}{133}\right)$
3. $J_3 = \left(\frac{66.5}{133}, \frac{69}{133}\right)$ ($|J_2| = |J_3|$)
4. $J_4 = \left(\frac{72}{133}, \frac{74}{133}\right)$ ($|J_1| = |J_4|$)
5. $J_5 = \left(\frac{78}{133}, \frac{79}{133}\right)$ ($|J_5| = 20$)

The following are the only students who are allowed.

$e(1, 5, 5)$.

$e(2, 4, 5)$,

$e(3, 4, 5)$.

$e(4, 4, 4)$.

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x

$e(2, 4, 5)$. Let the number of such students be y_1

$e(3, 4, 5)$. Let the number of such students be y_2 .

$e(4, 4, 4)$. Let the number of such students be z .

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x

$e(2, 4, 5)$. Let the number of such students be y_1

$e(3, 4, 5)$. Let the number of such students be y_2 .

$e(4, 4, 4)$. Let the number of such students be z .

$$1) |J_2| = |J_3|,$$

only students using J_2 are $e(2, 4, 5)$ – they use one share each,

only students using J_3 are $e(3, 4, 5)$ – they use one share each.

Hence $y_1 = y_2$. We call them both y .

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x

$e(2, 4, 5)$. Let the number of such students be y_1

$e(3, 4, 5)$. Let the number of such students be y_2 .

$e(4, 4, 4)$. Let the number of such students be z .

$$1) |J_2| = |J_3|,$$

only students using J_2 are $e(2, 4, 5)$ – they use one share each,

only students using J_3 are $e(3, 4, 5)$ – they use one share each.

Hence $y_1 = y_2$. We call them both y .

$$2) \text{ Since } |J_1| = |J_4|, x = 2y + 3z.$$

GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be x

$e(2, 4, 5)$. Let the number of such students be y_1

$e(3, 4, 5)$. Let the number of such students be y_2 .

$e(4, 4, 4)$. Let the number of such students be z .

1) $|J_2| = |J_3|,$

only students using J_2 are $e(2, 4, 5)$ – they use one share each,

only students using J_3 are $e(3, 4, 5)$ – they use one share each.

Hence $y_1 = y_2$. We call them both y .

2) Since $|J_1| = |J_4|,$ $x = 2y + 3z$.

3) Since $s_3 = 14,$ $x + 2y + z = 14$.

$$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}.$$

Contradiction.

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors

$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, m_1 muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, m_2 muffins of this type.

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors

$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, m_1 muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, m_2 muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$

$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, s_1 students of this type.

$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, s_2 students of this type.

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors

$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, m_1 muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, m_2 muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$

$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, s_1 students of this type.

$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, s_2 students of this type.

4) **Set up equations:**

$$m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)$$

$$m_1 + m_2 = 5$$

$$s_1 + s_2 = 3$$

MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors

$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, m_1 muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, m_2 muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$

$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, s_1 students of this type.

$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, s_2 students of this type.

4) **Set up equations:**

$$m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)$$

$$m_1 + m_2 = 5$$

$$s_1 + s_2 = 3$$

Natural Number Solution: $m_1 = 1, m_2 = 4, s_1 = 2, s_2 = 1$

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \dots, \frac{b-a}{b}$

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

- 1) **Guess** that the only piece sizes are $\frac{a}{b}, \dots, \frac{b-a}{b}$
- 2) **Muffin**=pieces add to 1: Vectors \vec{v}_i . x types.
 m_i muffins of type \vec{v}_i

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

- 1) **Guess** that the only piece sizes are $\frac{a}{b}, \dots, \frac{b-a}{b}$
- 2) **Muffin**=pieces add to 1: Vectors \vec{v}_i . x types.
 m_i muffins of type \vec{v}_i
- 3) **Student**=pieces add to $\frac{m}{s}$: Vectors \vec{u}_j . y types.
 s_j students of type \vec{u}_j

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

- 1) **Guess** that the only piece sizes are $\frac{a}{b}, \dots, \frac{b-a}{b}$
- 2) **Muffin**=pieces add to 1: Vectors \vec{v}_i . x types.
 m_i muffins of type \vec{v}_i
- 3) **Student**=pieces add to $\frac{m}{s}$: Vectors \vec{u}_j . y types.
 s_j students of type \vec{u}_j

4) **Set up equations:**

$$m_1 \vec{v}_1 + \dots + m_x \vec{v}_x = s_1 \vec{u}_1 + \dots + s_y \vec{u}_y$$

$$m_1 + \dots + m_x = m$$

$$s_1 + \dots + s_y = s$$

MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

- 1) **Guess** that the only piece sizes are $\frac{a}{b}, \dots, \frac{b-a}{b}$
- 2) **Muffin**=pieces add to 1: Vectors \vec{v}_i . x types.
 m_i muffins of type \vec{v}_i
- 3) **Student**=pieces add to $\frac{m}{s}$: Vectors \vec{u}_j . y types.
 s_j students of type \vec{u}_j
- 4) **Set up equations:**
$$m_1 \vec{v}_1 + \dots + m_x \vec{v}_x = s_1 \vec{u}_1 + \dots + s_y \vec{u}_y$$
$$m_1 + \dots + m_x = m$$
$$s_1 + \dots + s_y = s$$
- 5) **Look for Nat Numb sol.** If find can translate into procedure.

Later Results by Other People

1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input m, s , found $f(m, s)$ and the procedure REALLY FAST.
2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob's code is also REALLY FAST.
3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott's algorithm was fast (poly in m, s).
4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arXiv.
5. One corollary of the work: $f(m, s)$ only depends on m/s .

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.
- ▶ He brought to the seminar 11 muffins:
1 cut $(\frac{15}{30}, \frac{15}{30})$, 2 cut $(\frac{14}{30}, \frac{16}{30})$, 8 cut $(\frac{13}{30}, \frac{17}{30})$.

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.
- ▶ He brought to the seminar 11 muffins:
1 cut $(\frac{15}{30}, \frac{15}{30})$, 2 cut $(\frac{14}{30}, \frac{16}{30})$, 8 cut $(\frac{13}{30}, \frac{17}{30})$.
The five of us took pieces so we each got $\frac{11}{5}$ muffins.

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.
- ▶ He brought to the seminar 11 muffins:
1 cut $(\frac{15}{30}, \frac{15}{30})$, 2 cut $(\frac{14}{30}, \frac{16}{30})$, 8 cut $(\frac{13}{30}, \frac{17}{30})$.
The five of us took pieces so we each got $\frac{11}{5}$ muffins.
- ▶ He does a Bike-For-Food Charity. I asked him if I should give \$40.00 a year OR my Royalties. He chose the \$40.00.

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.
- ▶ He brought to the seminar 11 muffins:
1 cut $(\frac{15}{30}, \frac{15}{30})$, 2 cut $(\frac{14}{30}, \frac{16}{30})$, 8 cut $(\frac{13}{30}, \frac{17}{30})$.
The five of us took pieces so we each got $\frac{11}{5}$ muffins.
- ▶ He does a Bike-For-Food Charity. I asked him if I should give \$40.00 a year OR my Royalties. He chose the \$40.00.
First Year Royalties: \$40.00. The break-even point!

I meet Alan Frank!

I emailed Alan Frank, the **creator** of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

- ▶ He was delighted that his innocent problem, that he viewed as recreational, has led to so much math of interest.
- ▶ He brought to the seminar 11 muffins:
1 cut $(\frac{15}{30}, \frac{15}{30})$, 2 cut $(\frac{14}{30}, \frac{16}{30})$, 8 cut $(\frac{13}{30}, \frac{17}{30})$.
The five of us took pieces so we each got $\frac{11}{5}$ muffins.
- ▶ He does a Bike-For-Food Charity. I asked him if I should give \$40.00 a year OR my Royalties. He chose the \$40.00.
First Year Royalties: \$40.00. The break-even point!
Second Year Royalties: \$50.00. I'm up by \$10.00. Wow!