Chapter 1

\[ m \geq s \text{ then } f(m, s) \geq 1/3 \]

In this chapter we show that if \( m \geq s \), then \( f(m, s) \geq \frac{1}{3} \).

1.1 Example: \( f(19, 17) \geq \frac{1}{3} \)

We express \( \frac{19}{17} \) as \( \frac{57}{51} \) since other fractions will have a denominator of 51.

We initially divide all 19 muffins \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). There are now 57 pieces \( \frac{1}{3} \)-pieces. Since

\[ \frac{1}{3} \times 3 < \frac{19}{17} < \frac{1}{3} \times 4 \]

- The max number of pieces someone can get and have \( < \frac{19}{17} \) is 3.
- The min number of pieces someone can get and have \( > \frac{19}{17} \) is 4.

Hence we will give everyone either 3 or 4 \( \frac{1}{3} \)-pieces (which we will denote by \( W = 3 \) in the general technique). The only way to distribute 57 pieces so that everyone gets 3 or 4 pieces is to give 11 students 3 pieces and 6 students 4 pieces \((s_W = s_3 = 11 \text{ and } s_{W+1} = s_4 = 6 \text{ in the general technique})\). As usual a student who gets 3 (4) shares is called a 3-student (4-student).

We describe a process whereby students give pieces of muffins, called gifts, to other students so that, in the end, all students
have \( \frac{57}{51} \). Each gift leads to a change in how the muffins are cut in the first place; however, there will never be a muffin of size \(< \frac{1}{3}\).

Each 4-student has \( \frac{4}{3} = \frac{68}{51} \) and hence has to give (perhaps in several increments) \( \frac{68}{51} - \frac{57}{51} = \frac{11}{51} \) to get down to \( \frac{57}{51} \). Realize that if a 4-student gives \( \frac{11}{51} \) to a 3-student, then the 3-student now has \( \frac{51}{51} + \frac{11}{51} = \frac{62}{51} > \frac{57}{51} \).

Each 3-student has \( \frac{51}{51} \) and hence has to receive \( \frac{57}{51} - \frac{51}{51} = \frac{6}{51} \) to get up to \( \frac{57}{51} \).

Call the 11 3-students \( g_1, \ldots, g_{11} \).

Call the 6 4-students \( f_1, \ldots, f_6 \).

**Notation 1.1.** \( x(f_1 \to g_1) \) means the following: \( f_1 \) gives \( x \) to \( g_1 \) by taking two \( \frac{1}{3} \)-pieces, combining them, cutting off a piece of size \( x \), giving it to \( g_1 \) while keeping the rest. \( g_1 \) takes the piece given to him and combines it with a \( \frac{1}{3} \) piece. Notice that in terms of pieces we are taking three pieces of size \( \frac{1}{3} \) (2 from \( f_1 \) and 1 from \( g_1 \)) and turning them into 1 piece of size \( \frac{2}{3} - x \) and one of size \( \frac{1}{3} + x \). Hence we can easily rearrange how the muffins are cut.

We need to make sure this procedure never results in a piece that is \(< \frac{1}{3} \). In the above example (1) \( f_1 \) now has a piece of size \( \frac{2}{3} - x \), hence we need \( x \leq \frac{1}{3} \), (2) \( g_1 \) now has a piece of size \( \frac{1}{3} + x \), which is clearly \( \geq \frac{1}{3} \). Hence the only restriction is \( x \leq \frac{1}{3} \).

\begin{enumerate}
    \item \( \frac{11}{51}(f_1 \to g_1) \). Now \( f_1 \) has \( \frac{57}{51} \). YEAH. However, \( g_1 \) has \( \frac{62}{51} \).
    \item \( \frac{5}{51}(g_1 \to g_2) \). Now \( g_1 \) has \( \frac{51}{51} - \frac{5}{51} = \frac{57}{51} \). YEAH. However, \( g_2 \) has \( \frac{51}{51} + \frac{5}{51} = \frac{56}{51} \).
    \item \( \frac{1}{51}(f_2 \to g_2) \). Now \( g_2 \) has \( \frac{57}{51} \). YEAH. However, \( f_2 \) has \( \frac{67}{51} \).
    \item \( \frac{10}{51}(f_2 \to g_3) \). Now \( f_2 \) has \( \frac{57}{51} \). YEAH. However, \( g_3 \) has \( \frac{61}{51} \).
    \item \( \frac{4}{51}(g_3 \to g_4) \). Now \( g_3 \) has \( \frac{57}{51} \). YEAH. However, \( g_4 \) has \( \frac{55}{51} \).
    \item \( \frac{2}{51}(f_3 \to g_4) \). Now \( g_4 \) has \( \frac{57}{51} \). YEAH. However, \( f_3 \) has \( \frac{66}{51} \).
\end{enumerate}
(7) $\frac{6}{51}(f_3 \to g_3)$. Now $f_3$ has $\frac{57}{51}$. YEAH. However, $g_5$ has $\frac{60}{51}$.

(8) $\frac{3}{51}(g_5 \to g_6)$. Now $g_5$ has $\frac{57}{51}$. YEAH. However, $g_6$ has $\frac{54}{51}$.

(9) $\frac{4}{51}(f_4 \to g_6)$. Now $g_6$ has $\frac{57}{51}$. YEAH. However, $f_4$ has $\frac{65}{51}$.

(10) $\frac{4}{51}(f_4 \to g_7)$. Now $f_4$ has $\frac{57}{51}$. YEAH. However, $g_7$ has $\frac{57}{51}$.

(11) $\frac{2}{51}(g_7 \to g_8)$. Now $g_7$ has $\frac{57}{51}$. YEAH. However, $g_8$ has $\frac{54}{51}$.

(12) $\frac{1}{51}(f_5 \to g_8)$. Now $f_5$ has $\frac{57}{51}$. YEAH. However, $g_8$ has $\frac{58}{51}$.

(13) $\frac{7}{51}(f_5 \to g_9)$. Now $f_5$ has $\frac{57}{51}$. YEAH. However, $g_9$ has $\frac{58}{51}$.

(14) $\frac{1}{51}(g_9 \to g_{10})$. Now $g_9$ has $\frac{58}{51}$. YEAH. However, $g_{10}$ has $\frac{54}{51}$.

(15) $\frac{5}{51}(f_6 \to g_{10})$. Now $g_{10}$ has $\frac{57}{51}$. YEAH. However, $f_6$ has $\frac{63}{51}$.

(16) $\frac{6}{51}(f_6 \to g_{11})$. Now $f_6$ has $\frac{57}{51}$. YEAH. However, $g_{11}$ has $\frac{57}{51}$.

OH. thats a good thing!

YEAH- we are done.

Note that the first $x$ was $\frac{11}{51} \leq \frac{1}{3}$ and the remaining $x$ were all $\leq \frac{11}{51} \leq \frac{1}{3}$. Hence all pieces in the final procedure are $\geq \frac{1}{3}$.

End of Example

**Theorem 1.2.** For all $m \geq s$, $f(m, s) \geq \frac{1}{3}$.

**Proof.** Divide all the muffins into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let $W$ be such that

$$\frac{1}{3} \times W \leq \frac{m}{s} \leq \frac{1}{3}(W + 1).$$

Give some students $W \frac{1}{3}$-pieces and some $(W + 1) \frac{1}{3}$-pieces. How many students? Let $s_W (s_{W+1})$ be the number of students who get $W (W + 1) \frac{1}{3}$-pieces. Then:

$$W s_W + (W + 1) s_{W+1} = 3m$$

$$s_W + s_{W+1} = s$$

These equations have a unique solution and unique value of $W$ if $s$ does not divide $3m$. If $s$ does divide $3m$ there will be more than one possible value of $W$; however, we can pick arbitrarily. So we give $s_W$ students $W \frac{1}{3}$-pieces and $s_{W+1}$ students $W + 1 \frac{1}{3}$-pieces.
By the definition of $W$:

$$0 \leq \frac{m}{s} - \frac{W}{3} \leq \frac{1}{3} \quad (1.1)$$

$$0 \leq \frac{W + 1}{3} - \frac{m}{s} \leq \frac{1}{3} \quad (1.2)$$

Now we will need to smooth out the distribution so that everyone receives $\frac{m}{s}$. We will do this by a sequence of moves of the form $x(f_i \to g_j)$ or $x(g_i \to g_j)$, as defined in the example.

We will assume $s_{W+1}$ and $s_W$ are relatively prime (this only comes up in Claim 3 below). This is fine because if they have a common factor $d$, we can just use the procedure for the $\frac{s_{W+1}}{d}$, $\frac{s_W}{d}$ case repeated $d$ times.

Call the $s_W W$-students $g_1, \ldots, g_{s_W}$.

Call the $s_{W+1} (W+1)$-students $f_1, \ldots, f_{s_{W+1}}$.

Claim 1:

1. If $s_{W+1} < s_W$ then $\frac{W + 1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.
2. If $s_W < s_{W+1}$ then $\frac{W + 1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

Proof of Claim 1:

$$s_{W+1} \times \frac{W + 1}{3} + s_W \times \frac{W}{3} = m$$

$$s_{W+1} \times \left(\frac{m}{s} + \frac{W + 1}{3} - \frac{m}{s}\right) + s_W \left(\frac{m}{s} + \frac{W}{3} - \frac{m}{s}\right) = m$$

$$\left(s_{W+1} + s_W\right) \frac{m}{s} + s_{W+1} \left(\frac{W + 1}{3} - \frac{m}{s}\right) + s_W \left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$s \times \frac{m}{s} + s_{W+1} \left(\frac{W + 1}{3} - \frac{m}{s}\right) + s_W \left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$\frac{W + 1}{3} - \frac{m}{s} = \frac{s_W}{s_{W+1}} \left(\frac{m}{s} - \frac{W}{3}\right)$$
\[ m \geq s \quad \text{then} \quad f(m, s) \geq 1/3 \]

Both parts follow.

\textbf{End of Proof of Claim 1}

We give the procedure to obtain \( f(m, s) \leq \frac{1}{3} \). There are two cases.

\textbf{Case 1:} \( s_{W+1} < s_W \). Hence by Claim 1 \( \frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3} \).

1. Let \( x = \frac{W+1}{3} - \frac{m}{s} \). Note that \( x \leq \frac{1}{3} \). Do \( f_1 \rightarrow g_1 \). Now \( f_1 \) has \( \frac{m}{s} \). YEAH. However, \( g_1 \) has \( \frac{W}{3} + \frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} \).
   (This is where we use \( s_{W+1} < s_W \), or more accurately the consequence of that from Claim 1.)

2. Let \( x = \frac{2W+1}{3} - 2 \times \frac{m}{s} \). Do \( g_1 \rightarrow g_2 \). Now \( g_1 \) has \( \frac{m}{s} \). YEAH.

3. If \( g_2 \) has \( > \frac{m}{s} \) then \( g_2 \) gives enough to \( g_3 \) so that \( g_2 \) has \( \frac{m}{s} \).
   Keep up this chain of \( g_1, g_2, g_3, \ldots \) until there is a \( g_i \) such that \( g_i \) end up with \( < \frac{m}{s} \) (though more than the \( \frac{W}{3} \) that \( g_i \) had originally). This happens because \( g_{i-1} \) gives \( g_i \) what it can, so \( g_{i-1} \) ends with exactly \( \frac{m}{s} \), but its just not enough for \( g_i \) to have \( \frac{m}{s} \) as well :-(.

4. Do \( f_2 \rightarrow g_i \) where \( x \) is such that \( g_i \) will now have \( \frac{m}{s} \).

5. Do \( f_2 \rightarrow g_{i+1} \) where \( x \) is such that \( f_2 \) will now have \( \frac{m}{s} \).
   Repeat the same chain of \( g_i \)'s as in step 3.

6. Repeat the above steps until you are done.

We need to show that (1) there is never a piece of size \( < \frac{1}{3} \), and (2) the process ends with every student getting \( \frac{m}{s} \).

\textbf{Claim 2:} The first gift is \( \leq \frac{1}{3} \) and no gift is larger.

\textbf{Proof of Claim 2:} Let \( C = \frac{W+1}{3} - \frac{m}{s} \) which is the size of the first gift. By equation (2) \( C \leq \frac{1}{3} \).

Assume that all gifts so far have been \( \leq C \). We analyze the three kinds of gifts and show that in all cases the gift is \( \leq C \).

- \( x(f_1 \rightarrow g_j) \) where (1) initially \( f_i \) has \( > \frac{m}{s} \), \( g_j \) has \( < \frac{m}{s} \), and (2) after the gift \( f_i \) has \( \frac{m}{s} \). When this occurs it is \( f_i \)'s first or second gift giving. (This happens in steps 1 and 5 above, and later as well.) Before the gift \( f_i \) has at least \( \frac{m}{s} \) but at
most $\frac{W+1}{3}$, so this gift has size at most $\frac{W+1}{3} - \frac{m}{s} = C$.

- $x(g_i \rightarrow g_{i+1})$ where (1) initially $g_i$ has $> \frac{m}{s}$, $g_j$ has $< \frac{m}{s}$, and (2) after the gift $g_i$ has $\frac{m}{s}$. When this occurs, $g_i$ has received a gift once and this is $g_i$’s first time giving. (This happens in steps 2 and in the chain referred to in step 5.) Since $g_i$ just received a gift of size $\leq C$ she has $\leq \frac{W}{3} + C$. Hence the gift is $\leq \frac{W}{3} - \frac{m}{s} + C \leq C$.

- $x(f_i \rightarrow g_j)$ where (1) initially $f_i$ has $> \frac{m}{s}$, $g_j$ has $< \frac{m}{s}$, and (2) after the gift $g_j$ has $\frac{m}{s}$. This will be $f_i$’s first time giving. (This happens in step 4 above.) Before the gift $f_i$ has at least $\frac{W}{3}$ but at most $\frac{m}{s}$, so this gift has size at most $\frac{m}{s} - \frac{W}{3} \leq C$ (by Claim 1).

**Claim 3:** If $s_W$ and $s_{W+1}$ are relatively prime then the process terminates with all students having $\frac{m}{s}$.

**Proof of Claim 3:**

In each step all of the $f_i$ have at least $\frac{m}{s}$. In each step the number of students who have the correct amount of muffin goes up. One may be worried that at some point we will try to do step 4 (for example) of the procedure and there will be no $g_i$ left who need more muffin. But this is not possible because until the process terminates the $f$’s always have more muffins than they need, so there is always a $g$ with less muffins than they need.

One may also be worried that eventually we will get all of the $f$’s to have $\frac{m}{s}$, but the $g$’s will not all have $\frac{m}{s}$. This is not possible either, because whenever we only make gifts from $f$ to $g$, there is no $g$ with more than $\frac{m}{s}$.

Finally, if $s_W$ and $s_{W+1}$ are not relatively prime, it is possible that the procedure will terminate early because in step 5 the size of the donation $x$ is 0. If this occurred it would mean that there is some subset of $F$ $f$’s and $G$ $g$’s each of which has exactly $\frac{m}{s}$, and only made donations amongst themselves. But then $\frac{\mathcal{F}}{\mathcal{G}} = \frac{s_{W+1}}{s_W}$, a contradiction.

**End of Proof of Claim 3**
\[ m \geq s \Rightarrow f(m, s) \geq \frac{1}{3} \]

Case 2: \( s_W < s_{W+1} \). This is similar to Case 1 except that instead of \( f_1 \) giving \( g_1 \) so that \( f_1 \) has \( \frac{m}{s} \), \( f_1 \) gives to \( g_1 \) so that \( g_1 \) has \( \frac{m}{s} \). Hence we have a chain of \( f_i \)'s instead of a chain of \( g_i \)'s.

1.2 Conjectures About Extensions

We first restate the main theorem:

**Theorem 1.3.** For all \( m \geq s \), if \( V \geq 3 \) then \( f(m, s) \geq \frac{1}{3} \).

What if \( V = 4 \)? \( V = 5 \)?

**Conjecture 1.4.** There exists a function \( a(V) \) such that the following is true: For all \( m \geq s \), if \( V \geq V \) then \( f(m, s) \geq a(V) \).

What might \( a(V) \) look like? We know that \( a(3) = \frac{1}{3} \) and empirically it seems that \( \lim_{V \to \infty} a(V) = \frac{1}{2} \). One candidate is

\[
a(V) = \frac{V + 1}{2V + 6}
\]