Quantum Bits, Entanglement, and the CHSH Game

Exposition by William Gasarch and Evan Golub

July 29, 2024
1. We describe what qubits are mathematically and how they can be used. We ignore the Physics. Physicists really can create qubits that behave as we describe.

2. We describe what entangled qubits are mathematically and how they can be used. We ignore the Physics. Physicists really can create entangled qubits that behave as we describe.

3. We describe the CHSH game.

4. We give a strategy for the CHSH game where (1) the 2 players are classical, and (2) the prob of winning is $0.75$. We note that one can prove this is the best two players can do.

5. We give a strategy for the CHSH game where (1) the 2 players have qubits that are entangled, and (2) the prob of winning is larger than $0.75$. 
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Quantum Bits I: Measure Once

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Math Needed

Def

1. Let \( N \) be the following function on vectors of complex numbers:
\[
N(\alpha, \beta) = \alpha^2 + \beta^2.
\]
Note that \( N(\alpha, \beta) \) is the square of length of the vector \((\alpha, \beta)\).

For the rest of these slides we will assume that \( N \) is applied to pairs of reals. We note that the use of complex numbers is very important for quantum mechanics.

2. A \( 2 \times 2 \) matrix \( M \) is unitary if when \( Mv = u \),
\[
N(Mv) = N(u).
\]
So \( N \) preserves length.

Example
Let \( 0 \leq \theta \leq 2\pi \). The following matrix is unitary.
\[
M_{\theta} = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]
This matrix rotates vectors by \( \theta \).

On next slide we show that \( M_{\theta} \) is unitary.
Math Needed

Def

1. Let $N$ be the following function on vectors of complex numbers: $N(\alpha, \beta) = \alpha^2 + \beta^2$. Note that $N(\alpha, \beta)$ is the square of length of the vector $(\alpha, \beta)$. 

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2. A $2 \times 2$ matrix $M$ is unitary if when $Mv = u$, $N(Mv) = N(u)$. So $N$ preserves length.

Example

Let $0 \leq \theta \leq 2\pi$. The following matrix is unitary.

$$M_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This matrix rotates vectors by $\theta$.

On next slide we show that $M_\theta$ is unitary.
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This matrix rotates vectors by $\theta$. On next slide we show that $M_{\theta}$ is unitary.
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This matrix rotates vectors by $\theta$.

On next slide we show that $M_\theta$ is unitary.
Proof that $M_{\theta}$ is Unitary

Let $v = (\alpha, \beta)$ be a vector. We show $N(M_{\theta}(v)) = N(v)$.

$$
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
=
\begin{pmatrix}
\cos(\theta)\alpha - \sin(\theta)\beta \\
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$N(\cos(\theta) \alpha - \sin(\theta) \beta, \sin(\theta) \alpha + \cos(\theta) \beta)$
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Let $\mathbf{v} = (\alpha, \beta)$ be a vector. We show $N(M_\theta(\mathbf{v})) = N(\mathbf{v})$.

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$$

$$
= \cos^2(\theta)\alpha^2 + \sin^2(\theta)\alpha^2 + \sin^2(\theta)\beta^2 + \cos^2(\theta)\beta^2
$$

$$
= (\cos^2(\theta) + \sin^2(\theta))\alpha^2 + (\cos^2(\theta) + \sin^2(\theta))\beta^2
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$= \cos^2(\theta)\alpha^2 + \sin^2(\theta)\alpha^2 + \sin^2(\theta)\beta^2 + \cos^2(\theta)\beta^2$

$= (\cos^2(\theta) + \sin^2(\theta))\alpha^2 + (\cos^2(\theta) + \sin^2(\theta))\beta^2$

$= \alpha^2 + \beta^2$
Quantum Bits

**Def** A **qubit** is an ordered pair $(\alpha, \beta)$ such that $\alpha^2 + \beta^2 = 1$. ROUGHLY speaking it says that the prob that the bit is 0 is $\alpha^2$ and the prob the bit is 1 is $\beta^2$. 

Caveat A qubit can be measured in many ways:

1) Measuring the qubit in the standard basis is as above.
2) Measure the qubit in a different basis. We prefer to say that we multiply it by a unitary matrix and measure the output of that multiplication. So we say we measure $M_{\theta}(v)$. We will elaborate on this on the next slide.
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   \[
   \text{We measure } M_\theta(v).
   \]

We will elaborate on this on the next slide.
Measuring a Quantum Bit

If Alice has a qubit $v = (\alpha, \beta)$ she could do the following:

1) Measure it in the standard basis. This means that (1) she will get 0 with prob $\alpha^2$ and (2) she will get 1 with prob $\beta^2$.

2) Measure it in basis $\theta$. First compute $M_\theta(v) = w = (\gamma, \delta)$ where $\gamma^2 + \beta^2 = 1$. Now measure $w$. She will get 0 with prob $\gamma^2$ and 1 with prob $\delta^2$. This is referred to as measuring the qubit in a different basis or in a different frame.
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If Alice has a qubit \( \nu = (\alpha, \beta) \) she could do the following:

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2) **Measure it in basis \( \theta \)** First compute \( M_\theta(\nu) = w = (\gamma, \delta) \) where \( \gamma^2 + \beta^2 = 1 \). **Now** measure \( w \). She will get 0 with prob \( \gamma^2 \) and 1 with prob \( \delta^2 \).
Measuring a Quantum Bit

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This is referred to as **measuring the qubit in a different basis** or in a **different frame**.
Example

Alice has qubit \( v = (\alpha, \beta) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).
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2) If instead she measures $M_{\frac{\pi}{6}}(v)$ then we’ll see what happens.
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2) If instead she measures \( M_{\frac{\pi}{6}}(v) \) then we’ll see what happens.

Next two slides have the first and second coordinate of \( M_{\frac{\pi}{6}}(v) \)
Example (cont)

**First coordinate** of \( M_{\pi/6}(v) \) is

\[
\cos(\theta) \alpha - \sin(\theta) \beta = \cos(\frac{\pi}{6}) \frac{1}{\sqrt{2}} - \sin(\frac{\pi}{6}) \frac{1}{\sqrt{2}}
\]

\[
= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} - \frac{1}{2} \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.
\]
Example (cont)

**First coordinate** of $M_{\frac{\pi}{6}}(v)$ is

$$\cos(\theta)\alpha - \sin(\theta)\beta = \cos\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}} - \sin\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

**Note** \(\left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2 = \frac{4 - 2\sqrt{3}}{8} \approx 0.067\)
First coordinate of $M_{\pi/6}(v)$ is
\[
\cos(\theta)\alpha - \sin(\theta)\beta = \cos\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}} - \sin\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}}
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= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} - \frac{1}{2} \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.
\]

Note \((\frac{\sqrt{3}-1}{2\sqrt{2}})^2 = \frac{4-2\sqrt{3}}{8} \sim 0.067\)

Second coordinate of $M_{\pi/6}(v)$ is
\[
\sin(\theta)\alpha + \cos(\theta)\beta = \sin\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}} + \cos\left(\frac{\pi}{6}\right) \frac{1}{\sqrt{2}}
\]
Example (cont)

First coordinate of $M_{\pi/6}(v)$ is
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Second coordinate of $M_{\pi/6}(v)$ is
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= \frac{1}{2} \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}.
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Example (cont)

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Note \( \left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{4 + 2\sqrt{3}}{8} \sim 0.933 \)
Alice has qubit $v = (\alpha, \beta) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. 

1. If she measures $v$ in the standard basis then $\Pr(0) = \frac{1}{2}$, $\Pr(1) = \frac{1}{2}$.

2. If instead she measures $w = M_{\pi/6}(v)$ then $\Pr(0) \sim 0.067$, $\Pr(1) \sim 0.933$. A rotation of 0 gave $\Pr(0) = 0.5$, whereas a rotation of $\pi/6$ made $\Pr(0) = 0.067$ which is much smaller. How does $\theta$ affect $\Pr(0)$ as $0 \leq \theta \leq \pi/4$, $\Pr(0)$ goes from $\frac{1}{2}$ to 0.

The next few slides investigate this issue further.
Upshot of Example

Alice has qubit \( v = (\alpha, \beta) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \)

1. If she measures \( v \) in the standard basis then
   
   - \( \Pr(0) = \frac{1}{2} \)
   - \( \Pr(1) = \frac{1}{2} \).

2. If instead she measures \( w = M_{\pi/6}(v) \) then
   
   - \( \Pr(0) \approx 0.067 \)
   - \( \Pr(1) \approx 0.933 \).

A rotation of \( 0 \) gave \( \Pr(0) = 0.5 \), whereas a rotation of \( \pi/6 \) made \( \Pr(0) = 0.067 \) which is much smaller. How does \( \theta \) affect \( \Pr(0) \) as \( 0 \leq \theta \leq \pi/4 \), \( \Pr(0) \) goes from \( \frac{1}{2} \) to \( 0 \).

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   - $Pr(0) = \frac{1}{2}$
   - $Pr(1) = \frac{1}{2}$.

2. If instead she measures $w = M_{\frac{\pi}{6}}(v)$ then
   - $Pr(0) \sim 0.067$.
   - $Pr(1) \sim 0.933$.

A rotation of $0\degree$ gave $Pr(0) = 0.5$, whereas a rotation of $\frac{\pi}{6}\degree$ made $Pr(0) = 0.067$ which is much smaller. How does $\theta$ affect $Pr(0)$? As $0 \leq \theta \leq \frac{\pi}{4}$, $Pr(0)$ goes from $\frac{1}{2}$ to $0$. 
Alice has qubit \( v = (\alpha, \beta) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

1. If she measures \( v \) in the standard basis then
   - \( \text{Pr}(0) = \frac{1}{2} \)
   - \( \text{Pr}(1) = \frac{1}{2} \).

2. If instead she measures \( w = M_{\pi/6}(v) \) then
   - \( \text{Pr}(0) \sim 0.067 \).
   - \( \text{Pr}(1) \sim 0.933 \).

A rotation of 0 gave \( \text{Pr}(0) = 0.5 \), whereas a rotation of \( \frac{\pi}{6} \) made \( \text{Pr}(0) = 0.067 \) which is much smaller. How does \( \theta \) affect \( \text{Pr}(0) \)?

**as \( 0 \leq \theta \leq \frac{\pi}{4} \), \( \text{Pr}(0) \) goes from \( \frac{1}{2} \) to 0.**

The next few slides investigate this issue further.
How Does $\theta$ Affect $\text{Pr}(0)$?

Alice has the qubit $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.  

As $\theta$ gets bigger what happens?

1. For $0 \leq \theta \leq \frac{\pi}{4}$, $\text{Pr}(0)$ goes from $\frac{1}{2}$ to $0$.
2. For $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, $\text{Pr}(0)$ goes from $0$ to $\frac{1}{2}$.
3. For $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, $\text{Pr}(0)$ goes from $\frac{1}{2}$ to $1$.
4. For $\frac{3\pi}{4} \leq \theta \leq \pi$, $\text{Pr}(0)$ goes from $1$ to $\frac{1}{2}$.

The next few slides give actual numbers.
How Does $\theta$ Affect $\Pr(0)$?

Alice has the qubit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. She is going measure $w = M_{\theta}(v)$.
How Does $\theta$ Affect $\Pr(0)$?

Alice has the qubit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. She is going to measure $w = M_\theta(v)$.

$\theta = 0 : \Pr(0) = \frac{1}{2}$.

$\theta = \pi/60: \Pr(0) = 0.448$, close to $\frac{1}{2}$.
How Does $\theta$ Affect $\Pr(0)$?

Alice has the qubit $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
She is going measure $w = M_\theta(v)$
$\theta = 0 : \Pr(0) = \frac{1}{2}$.
$\theta = \pi/60: \Pr(0) = 0.448$, close to $\frac{1}{2}$.
As $\theta$ gets bigger what happens?
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$\theta = \pi/60 : \Pr(0) = 0.448$, close to $\frac{1}{2}$.

As $\theta$ gets bigger what happens?

1. For $0 \leq \theta \leq \frac{\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 0.
How Does $\theta$ Affect $\Pr(0)$?

Alice has the qubit $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
She is going to measure $w = M_\theta(v)$

$\theta = 0$: $\Pr(0) = \frac{1}{2}$.
$\theta = \pi/60$: $\Pr(0) = 0.448$, close to $\frac{1}{2}$.

As $\theta$ gets bigger what happens?

1. For $0 \leq \theta \leq \frac{\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 0.
2. For $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, $\Pr(0)$ goes from 0 to $\frac{1}{2}$.
How Does $\theta$ Affect $\text{Pr}(0)$?

Alice has the qubit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.
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3. For $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, $\text{Pr}(0)$ goes from $\frac{1}{2}$ to 1.
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Alice has the qubit $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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3. For $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 1.
4. For $\frac{3\pi}{4} \leq \theta \leq \pi$, $\Pr(0)$ goes from 1 to $\frac{1}{2}$.

The next few slides give actual numbers.
How Does $\theta$ Affect $\Pr(0)$?

Alice has the qubit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

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$\theta = 0 : \Pr(0) = \frac{1}{2}$.

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As $\theta$ gets bigger what happens?

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3. For $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 1.
4. For $\frac{3\pi}{4} \leq \theta \leq \pi$, $\Pr(0)$ goes from 1 to $\frac{1}{2}$.

The next few slides give actual numbers.
\[ 0 \leq \theta \leq \frac{\pi}{4} \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \text{Pr}(0) = \alpha^2 )</th>
<th>( \text{Pr}(1) = \beta^2 )</th>
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<td>0</td>
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<td>+0.71</td>
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<tr>
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\[ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \]

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\[ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4} \]

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<th>( \Pr(0) = \alpha^2 )</th>
<th>( \Pr(1) = \beta^2 )</th>
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\[ \frac{3\pi}{4} \leq \theta \leq \pi \]

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<td>-0.71</td>
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</table>
Alice has a qubit $v = (\alpha, \beta)$.

1) Alice measures $v$ in the standard basis and gets bit $b$.
2) Alice then measures $v$ in the standard basis again. She will get $b$. She cannot get anything else.

This is not weird. Here is a classical analog: Alice has a box that has a coin in it with sides labelled 0 and 1. She opens it and sees a face up. She closes it. She opens it again. She still sees a face up.
Alice has a qubit $\nu = (\alpha, \beta)$.

1) Alice measures $\nu$ in the standard basis and gets bit $b$. 

This is not weird. Here is a classical analog: Alice has a box that has a coin in it with sides labelled 0 and 1. She opens it and sees a $b$ face up. She closes it. She opens it again. She still sees a $b$. 

Alice has a qubit $\nu = (\alpha, \beta)$.

1) Alice measures $\nu$ in the standard basis and gets bit $b$.
2) Alice then measures $\nu$ in the standard basis again.
Alice has a qubit \( \nu = (\alpha, \beta) \).

1) Alice measures \( \nu \) in the standard basis and gets bit \( b \).
2) Alice then measures \( \nu \) in the standard basis again.

She will get \( b \).
Alice has a qubit \( v = (\alpha, \beta) \).

1) Alice measures \( v \) in the standard basis and gets bit \( b \).
2) Alice then measures \( v \) in the standard basis again. **She will get \( b \).** She cannot get anything else.
Measuring a Qubit Twice in the Standard Basis

Alice has a qubit $\nu = (\alpha, \beta)$.

1) Alice measures $\nu$ in the standard basis and gets bit $b$.
2) Alice then measures $\nu$ in the standard basis again. **She will get $b$.** She cannot get anything else.

This is **not** weird. Here is a classical analog:

Alice has a box that has a coin in it with sides labelled 0 and 1. She opens it and sees a $b$ face up. She closes it. She opens it again. She still sees a $b$. 
Alice has a qubit $\nu = (\alpha, \beta)$.

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Alice has a box that has a coin in it with sides labelled 0 and 1.
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1) Alice measures $\nu$ in the standard basis and gets bit $b$.
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This is **not** weird. Here is a classical analog:

Alice has a box that has a coin in it with sides labelled 0 and 1. She opens it and sees a $b$ face up.
Alice has a qubit $v = (\alpha, \beta)$.

1) Alice measures $v$ in the standard basis and gets bit $b$.
2) Alice then measures $v$ in the standard basis again. She will get $b$. She cannot get anything else.

This is not weird. Here is a classical analog:

Alice has a box that has a coin in it with sides labelled 0 and 1. She opens it and sees a $b$ face up. She closes it. She opens it again. She still sees a $b$. 
Measuring a Qubit in Standard Basis and Non-Standard Basis

1) Alice measures \( v \) in the standard basis and gets bit \( b \).

2) Alice then measures \( w = M_\theta(v) \).

The prob that she gets \( b \) is \( \cos^2(\theta) \).

3) Next slide generalizes this.

Not Weird. Alice shakes the box and maybe the coin flips over.

ASK A PHYSICS PERSON IF THIS IS A GOOD ANALOGY WHEN PRESENTING TO BEA.

Weird. \( \cos^2(\theta) \)?
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Not Weird. Alice shakes the box and maybe the coin flips over. Ask a physics person if this is a good analogy when presenting to BEA.

Weird. $\cos^2(\theta)$?
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1) Alice measures $w = M_{\theta_1}(\nu)$ and gets bit $b$. 
2) Alice then measures $w' = M_{\theta_2}(w)$. 

The prob that she gets $b$ is $\cos^2(\theta_1 - \theta_2)$. 

Not Weird. Alice shakes the box and maybe the coin flips over. 

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Measuring a Qubit in Two Different Basis

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2 People Measure a Qubit in Two Different Basis
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Example

Alice has qubit $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. 

1) Alice measures $v$ in the standard basis and gets 0.
2) Bob then measures $w = M_{\pi/6}(v)$. 

$\Pr(0) = \Pr(\text{Bob and Alice agree}) = \cos^2(0 - \pi/6) = 0.75.$ 

$\Pr(1) = 1 - \Pr(0) = 0.25.$
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Entanglement

Exposition by
William Gasarch and Evan Golub

July 29, 2024
We say what Alice and Bob can do if they have qubits that are entangled.
We say what Alice and Bob can do if they have qubits that are entangled.

We first describe four scenarios to contrast the case of 2 qubits that are entangled to other cases, both classical and quantum.
Four Scenarios

1) Charles picks a bit $b \in \{0,1\}$ at random. Charles gives Alice a box with $b$ in it, and Bob a box with the $b$ in it. If Alice opens the box and sees $b$, she knows that Bob's box also has $b$.

2) Charles gives Alice a qubit $(\alpha, \beta)$ and also gives Bob a different qubit $(\alpha, \beta)$. Alice and Bob both know $\alpha$ and $\beta$. If Alice measures in the standard basis and gets a 1 this will not help her know what Bob's measurement is.

3) Alice has a qubit. Alice measures it in the standard basis and gets 0. Alice then gives the qubit to Bob. He measures it in the standard basis. Alice knows that he will get a 0.

4) Alice has a qubit. Alice measures it in the standard basis and gets 0. Alice then gives the qubit to Bob. He multiplies it by $M_{\pi/6}$ and then measures it. $\Pr(0) = \cos^2(0 - \pi/6) = 0.75$, $\Pr(1) = 0.25$. So Alice thinks Bob will probably get a 0.
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4) Alice has a qubit. Alice measures it in the standard basis and gets 0. Alice then gives the qubit to Bob. He multiplies it by $M = \pi / 6$ and then measures it. $\Pr(0) = \cos^2(0 - \pi / 6) = 0.75$, $\Pr(1) = 0.25$. So Alice thinks Bob will probably get a 0.
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The Four Scenarios Are Not Weird Because

In Scenario 1 Alice and Bob are given the same bit, so not weird that they are the same.

In Scenario 2 Alice and Bob are given different qubits, so not weird that they are not correlated.

In Scenario 3 Alice and Bob are looking at the same qubit, so not weird they are the same.

In Scenario 4 Alice and Bob are looking at versions of the same qubit, so not weird that they are correlated (perhaps weird that it involves cosines).

Quantum Entanglement will be weird since Alice and Bob will have different qubits that are far apart, yet the measures will be correlated.
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Quantum Entanglement will be weird since Alice and Bob will have different qubits that are far apart, yet the measures will be correlated.
Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled.
Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice and Bob are light years apart.
Alice and Bob Have Entangled Qbits

Alice has a qubit \( \nu_A = (\alpha, \beta) \). Bob has a qubit \( \nu_B = (\alpha, \beta) \). \( \nu_A \) and \( \nu_B \) are entangled. Alice and Bob are light years apart.

1) If Alice measures \( \nu_A \) in the standard basis and gets \( b \) and then Bob measures \( \nu_B \) in the standard basis he will also get \( b \). Note that \( \nu_A \) and \( \nu_B \) are not the same qubit. In fact, they are far apart.

2) If Alice measures \( \nu_A \) in the standard basis and gets \( b \) and then Bob measures \( \nu_B \) in the standard basis \( \theta_B \) (\( \nu_B \)), and \( \theta_B \) is very close to 0, then Bob will probably get \( b \) as well (we will formalize this in the next point).

3) If Alice measures \( \nu_A \) in the standard basis and Bob measures \( \nu_B \) in the standard basis then the probability that they get the same answer is \( \cos^2(\theta_A - \theta_B) \).

Note: This is weird. The two entangled qubits are different and far apart yet they are somehow linked together.
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Note This is weird. The two entangled qubits are different and far apart yet they are somehow linked together.
Contrast Entangled and Not Entangled

Alice has a qubit $\nu_A = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. Bob has a qubit $\nu_B = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. 

1) Alice measures $\nu_A$ in the standard basis and gets 0.
2) Then Bob measures $\nu_B$.
   2a) If $\nu_A$ and $\nu_B$ are not entangled then $\Pr(\text{Bob gets 0}) \sim 0.067$ $\Pr(\text{Bob gets 1}) \sim 0.933$.
   2b) If $\nu_A$ and $\nu_B$ are entangled then $\Pr(\text{Bob gets 0}) = \Pr(\text{Alice & Bob agree}) = \cos^2 \left( \frac{\pi}{6} - 0 \right) \sim 0.75$.
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The CHSH Game

Exposition by
William Gasarch and Evan Golub

July 29, 2024
The CHSH Game

(CHSH stands for the authors of the paper this appeared in: John Clauser, Michael Horne, Abner Shimony, Richard Holt.)
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1. Charles sends Alice a bit $x$ and Bob a bit $y$. Both $x$ and $y$ were chosen uniformly random.
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1. Charles sends Alice a bit $x$ and Bob a bit $y$. Both $x$ and $y$ were chosen uniformly random.

If $x \land y = a \oplus b$ then Alice and Bob win. Else they lose.
The CHSH Game

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1. Charles sends Alice a bit $x$ and Bob a bit $y$. Both $x$ and $y$ were chosen uniformly random.
3. If $x \land y = a \oplus b$ then Alice and Bob win. Else they lose.
On the next few slides we discuss strategies with an eye towards asking how often they win.
All 0 Strategy

Since $x \land y$ is mostly 0, make $a \oplus b$ always 0, so a strong strategy is Alice and Bob to both send 0.
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Alice and Bob win with probability 0.75.
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Alice and Bob win with probability 0.75.
Mostly 0 Strategy

Since $x \land y$ is mostly 0 but not all the time we want Alice to sometimes send a 1.
Since $x \land y$ is mostly 0 but not all the time we want Alice to sometimes send a 1.

If Alice sees a 1 then with prob $p$ (to be determined) she sends a 1. Bob still always sends a 0.
Since $x \land y$ is mostly 0 but not all the time we want Alice to sometimes send a 1.

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Alice will flip a coin with sides 0 and 1, prob $p$ of getting a 1.
Mostly 0 Strategy

Since $x \land y$ is mostly 0 but not all the time we want Alice to sometimes send a 1.

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Alice will flip a coin with sides 0 and 1, prob $p$ of getting a 1.

Next Slide analyses the prob that they win.
Analyzing the Mostly 0 Strategy

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<th>x</th>
<th>y</th>
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<th>a</th>
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<th>Wins?</th>
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In the first four rows the coin flip is irrelevant. If \((x, y) = (1, 0)\) then they win if the coin is 0, so prob \(1 - p\). If \((x, y) = (1, 1)\) then they win if the coin is 1, so prob \(p\).

Hence they win when any of the following happen:

1) \((x, y) \in\{(0, 0), (0, 1)\}\). Thats prob \(\frac{1}{2}\).
2) \((x, y) = (1, 0)\) and the coin is 0. Thats prob \(\frac{1}{4} \times (1 - p)\).
3) \((x, y) = (1, 1)\) and the coin is 1. Thats prob \(\frac{1}{4} \times p\).

So the prob of winning is \(\frac{1}{2} + \frac{1}{4} - \frac{p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75\). No better.
Analyzing the Mostly 0 Strategy

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In the first four rows the coin flip is irrelevant.
Analyzing the Mostly $0$ Strategy

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In the first four rows the coin flip is irrelevant. If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$. 
Analyzing the Mostly 0 Strategy

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Analyzing the Mostly 0 Strategy

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Hence they win when any of the following happen:

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In the first four rows the coin flip is irrelevant. If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$. If $(x, y) = (1, 1)$ then they win if the coin is 1, so prob $p$.

Hence they win when any of the following happen:

1) $(x, y) \in \{(0, 0), (0, 1)\}$. Thats prob $\frac{1}{2}$. 

No better.
Analyzing the Mostly 0 Strategy

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If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$.
If $(x, y) = (1, 1)$ then they win if the coin is 1, so prob $p$.

Hence they win when any of the following happen:

1) $(x, y) \in \{(0, 0), (0, 1)\}$. Thats prob $\frac{1}{2}$.
2) $(x, y) = (1, 0)$ and the coin is 0. Thats prob $\frac{1}{4} \times (1 - p)$. 

No better.
Analyzing the Mostly 0 Strategy

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If \((x, y) = (1, 0)\) then they win if the coin is 0, so prob 1 - \(p\).
If \((x, y) = (1, 1)\) then they win if the coin is 1, so prob \(p\).

Hence they win when any of the following happen:
1) \((x, y) \in \{(0, 0), (0, 1)\}\). Thats prob \(\frac{1}{2}\).
2) \((x, y) = (1, 0)\) and the coin is 0. Thats prob \(\frac{1}{4} \times (1 - p)\).
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In the first four rows the coin flip is irrelevant.
If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$.
If $(x, y) = (1, 1)$ then they win if the coin is 1, so prob $p$.

Hence they win when any of the following happen:
1) $(x, y) \in \{(0, 0), (0, 1)\}$. That's prob $\frac{1}{2}$.
2) $(x, y) = (1, 0)$ and the coin is 0. That's prob $\frac{1}{4} \times (1 - p)$.
3) $(x, y) = (1, 1)$ and the coin is 1. That's prob $\frac{1}{4} \times p$.

So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$. 
Analyzing the Mostly 0 Strategy

<table>
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<th>$x$</th>
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<th>coin</th>
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<th>$x \land y$</th>
<th>$a \oplus b$</th>
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Hence they win when any of the following happen:
1) $(x, y) \in \{(0, 0), (0, 1)\}$. Thats prob $\frac{1}{2}$.
2) $(x, y) = (1, 0)$ and the coin is 0. Thats prob $\frac{1}{4} \times (1 - p)$.
3) $(x, y) = (1, 1)$ and the coin is 1. Thats prob $\frac{1}{4} \times p$.

So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$. No better.
Is There a Better Strategy?

The following are known:
Is There a Better Strategy?

The following are known:

1. There is no deterministic strategy that can win with probability more than 0.75.
Is There a Better Strategy?

The following are known:

1. There is no deterministic strategy that can win with probability more than 0.75.
2. There is no randomized strategy that can win with probability more than 0.75.
We will show on the next two slides that if
If Alice and Bob Have Entangled Qbits

We will show on the next two slides that if

1. Alice has a qubit $\nu_A = (\alpha, \beta)$,
If Alice and Bob Have Entangled Qbits

We will show on the next two slides that if

1. Alice has a qubit $v_A = (\alpha, \beta)$,
2. Bob has a qubit $v_B = (\alpha, \beta)$, and
If Alice and Bob Have Entangled Qbits

We will show on the next two slides that if

1. Alice has a qubit \( \nu_A = (\alpha, \beta) \),
2. Bob has a qubit \( \nu_B = (\alpha, \beta) \), and
3. \( \nu_A \) and \( \nu_B \) are entangled.
We will show on the next two slides that if

1. Alice has a qubit $v_A = (\alpha, \beta)$,
2. Bob has a qubit $v_B = (\alpha, \beta)$, and
3. $v_A$ and $v_B$ are entangled.

then Alice and Bob have a strategy that wins the CHSH game with probability $\frac{13}{16} = 0.8125 > 0.75$. 

If Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. 
Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $\nu_A = (\alpha, \beta)$. Bob has a qubit $\nu_B = (\alpha, \beta)$. $\nu_A$ and $\nu_B$ are entangled.

Alice gets $x$, Bob gets $y$.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $\nu_A = (\alpha, \beta)$. Bob has a qubit $\nu_B = (\alpha, \beta)$. $\nu_A$ and $\nu_B$ are entangled. Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{3}}(\nu)$. $a$ is result.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled.

Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{3}}(v)$. $a$ is result.
2. $x = 1$: Alice measures $v$ in the standard basis. $a$ is result.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $\nu_A = (\alpha, \beta)$. Bob has a qubit $\nu_B = (\alpha, \beta)$. $\nu_A$ and $\nu_B$ are entangled.
Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{3}}(\nu)$. $a$ is result.
2. $x = 1$: Alice measures $\nu$ in the standard basis. $a$ is result.
3. $y = 0$: Bob measures $M_{\frac{\pi}{6}}(\nu)$. $b$ is result.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\pi/3}(v)$. $a$ is result.
2. $x = 1$: Alice measures $v$ in the standard basis. $a$ is result.
3. $y = 0$: Bob measures $M_{\pi/6}(v)$. $b$ is result.
4. $y = 1$: Bob measures $M_{\pi/2}(v)$. $b$ is result.
If Alice and Bob Have Entangled Qbits

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

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4. $y = 1$: Bob measures $M_{\frac{\pi}{2}}(v)$. $b$ is result.

We analyze all four cases $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ on the next slides.
Each Scenario

Alice and Bob share the entangled bit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$. 

1. $(x, y) = (0, 0)$. Alice: $\pi/3$. Bob: $\pi/6$. Prob they agree: $\cos^2(\pi/3 - \pi/6) = \cos^2(\pi/6) = (\sqrt{3}/2)^2 = \frac{3}{4}$.

2. $(x, y) = (0, 1)$. Alice: $2\pi/3$. Bob: $\pi/2$. Prob they agree: $\cos^2(2\pi/3 - \pi/2) = \cos^2(\pi/6) = (\sqrt{3}/2)^2 = \frac{3}{4}$.

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\pi/6$. Prob they agree: $\cos^2(\pi/6 - 0) = \cos^2(\pi/6) = \frac{3}{4}$.

4. $(x, y) = (1, 1)$. Alice: 0. Bob: $\pi/2$. Prob they agree: $\cos^2(\pi/2 - 0) = \cos^2(\pi/2) = 0$.

So prob they do not agree is 1. Hence the prob of a win is $\frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times 1 = \frac{13}{16} = 0.8125$. 
Each Scenario

Alice and Bob share the entangled bit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\pi/3$. Bob: $\pi/6$. Prob they agree:
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   \]
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Alice and Bob share the entangled bit \( v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \).

1. \((x, y) = (0, 0)\). Alice: \(\frac{\pi}{3}\). Bob: \(\frac{\pi}{6}\). Prob they agree:
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   \cos^2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.
   \]

2. \((x, y) = (0, 1)\). Alice: \(\frac{2\pi}{3}\). Bob: \(\frac{\pi}{2}\). Prob they agree:
   \[
   \cos^2\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{6}\right) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}.
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Each Scenario

Alice and Bob share the entangled bit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{3}$. Bob: $\frac{\pi}{6}$. Prob they agree:
   $$\cos^2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right)(\frac{\sqrt{3}}{2})^2 = \frac{3}{4}.$$

2. $(x, y) = (0, 1)$. Alice: $\frac{2\pi}{3}$. Bob: $\frac{\pi}{2}$. Prob they agree:
   $$\cos^2\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{6}\right) = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}.$$

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{6}$. Prob they agree:
   $$\cos^2\left(\frac{\pi}{6} - 0\right) = \cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}.$$
Each Scenario

Alice and Bob share the entangled bit $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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   \cos^2\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.
   \]

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   \[
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2. $(x, y) = (0, 1)$. Alice: $\frac{2\pi}{3}$. Bob: $\frac{\pi}{2}$. Prob they agree:
   \[ \cos^2\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}. \]

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{6}$. Prob they agree:
   \[ \cos^2\left(\frac{\pi}{6} - 0\right) = \cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}. \]

4. $(x, y) = (1, 1)$. Alice: 0. Bob: $\frac{\pi}{2}$. Prob they agree:
   \[ \cos^2\left(\frac{\pi}{2} - 0\right) = \cos^2\left(\frac{\pi}{2}\right) = 0. \]
   So prob they do not agree is 1.
Alice and Bob share the entangled bit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{3}$. Bob: $\frac{\pi}{6}$. Prob they agree:
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2. $(x, y) = (0, 1)$. Alice: $\frac{2\pi}{3}$. Bob: $\frac{\pi}{2}$. Prob they agree:
   \[ \cos^2\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}. \]

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{6}$. Prob they agree:
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   So prob they do not agree is 1.

Hence the prob of a win is
\[ \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times 1 = \frac{13}{16} = 0.8125. \]
What Does This Mean?

1. Physicists have actually done this in the lab.
2. This is evidence that quantum mechanics is correct.
3. There are things we can do better in the quantum world than in the classical world.
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1. Physicists have actually done this in the lab.
2. This is evidence that quantum mechanics is correct.
3. There are things we can do better in the quantum world than in the classical world.
Can We Do Better?

We have shown the following:

\[
\text{If Alice has } v_A \text{ and Bob has } v_B \text{ and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125.}
\]
We have shown the following:

If Alice has $v_A$ and Bob has $v_B$ and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

Vote Which of the following is true:
Can We Do Better?

We have shown the following:

If Alice has $v_A$ and Bob has $v_B$ and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob $0.8125$

Vote Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is known.
Can We Do Better?

We have shown the following:

If Alice has $v_A$ and Bob has $v_B$ and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob $0.8125$

Vote Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is known.
2. If Alice has ... then the max prob that Alice and Bob can achieve is $0.8125$ and this is known.
We have shown the following:

**If Alice has** $v_A$ **and Bob has** $v_B$ **and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125**

**Vote** Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is *known*.
2. If Alice has ... then the max prob that Alice and Bob can achieve is 0.8125 and this is *known*.
3. The question of if Alice and Bob can do better than 0.8125 is *Unknown to Science*. 
We have shown the following:

**If Alice has** $v_A$ **and Bob has** $v_B$ **and those qubits are entangled then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125**

**Vote** Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with Prob $p > 0.8125$ and this is known.

2. If Alice has ... then the max prob that Alice and Bob can achieve is 0.8125 and this is known.

3. The question of if Alice and Bob can do better than 0.8125 is **Unknown to Science**.

Answer on the next slide.
If Alice and Bob Can Do Better than 0.8125

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. 
Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled.
If Alice and Bob Can Do Better than 0.8125

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$. 
If Alice and Bob Can Do Better than 0.8125

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{4}}(v)$. $a$ is result.
If Alice and Bob Can Do Better than 0.8125

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled.

Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{4}}(v)$. $a$ is result.

2. $x = 1$: Alice measures $v$ in the standard basis. $a$ is result.
Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

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If Alice and Bob Can Do Better than 0.8125

Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\frac{\pi}{4}}(v)$. $a$ is result.
2. $x = 1$: Alice measures $v$ in the standard basis. $a$ is result.
3. $y = 0$: Bob measures $M_{\frac{\pi}{8}}(v)$. $b$ is result.
4. $y = 1$: Bob measures $M_{\frac{3\pi}{8}}(v)$. $b$ is result.

We analyze all four cases $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ on the next slides.
Alice has a qubit $v_A = (\alpha, \beta)$. Bob has a qubit $v_B = (\alpha, \beta)$. $v_A$ and $v_B$ are entangled. Alice gets $x$, Bob gets $y$.

1. $x = 0$: Alice measures $M_{\pi/4}(v)$. $a$ is result.
2. $x = 1$: Alice measures $v$ in the standard basis. $a$ is result.
3. $y = 0$: Bob measures $M_{3\pi/8}(v)$. $b$ is result.
4. $y = 1$: Bob measures $M_{\pi/8}(v)$. $b$ is result.

We analyze all four cases $(x, y) \in \{(0,0), (0,1), (1,0), (1,1)\}$ on the next slides.
Each Scenario

Alice and Bob share the entangled bit \( v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).
Each Scenario

Alice and Bob share the entangled bit $\nu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob they agree:
   
   $\cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$. 

So prob they do not agree is $\sim 1 - 0.853 = 0.146$.

If $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$ then the prob $a = b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$.

If $(x, y) = (1, 1)$ then the prob $a \neq b$ is $\sim 0.853$, so prob of WIN is $\sim 0.146$.

Hence the prob of a win is $\sim \frac{3}{4} (0.853) + \frac{1}{4} (0.146) = 0.853$. Better than $0.8125$.

The exact prob of winning is $\cos^2\left(\frac{\pi}{8}\right)$. 
Each Scenario

Alice and Bob share the entangled bit \( \nu = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

1. \((x, y) = (0, 0)\). Alice: \( \frac{\pi}{4} \). Bob: \( \frac{\pi}{8} \). Prob they agree:
\[
\cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
\]

2. \((x, y) = (0, 1)\). Alice: \( \frac{\pi}{4} \). Bob: \( \frac{3\pi}{8} \). Prob they agree:
\[
\cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
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Each Scenario

Alice and Bob share the entangled bit \( \nu = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

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   \[ \cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853. \]

3. \((x, y) = (1, 0)\). Alice: 0. Bob: \(\frac{\pi}{8}\). Prob they agree: 
   \[ \cos\left(\frac{\pi}{8} - 0\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853. \]
Each Scenario

Alice and Bob share the entangled bit $\nu = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob they agree:
   \[
   \cos^2 \left( \frac{\pi}{4} - \frac{\pi}{8} \right) = \cos^2 \left( \frac{\pi}{8} \right) \sim 0.853.
   \]

2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob they agree:
   \[
   \cos^2 \left( \frac{3\pi}{8} - \frac{\pi}{4} \right) = \cos^2 \left( \frac{\pi}{8} \right) \sim 0.853.
   \]

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{8}$. Prob they agree:
   \[
   \cos \left( \frac{\pi}{8} - 0 \right) = \cos^2 \left( \frac{\pi}{8} \right) \sim 0.853.
   \]

4. $(x, y) = (1, 1)$. Alice: 0. Bob: $3\pi/8$. Prob they agree:
   \[
   \cos \left( \frac{3\pi}{8} - 0 \right) = \cos^2 \left( \frac{3\pi}{8} \right) \sim 0.146.
   \]
Each Scenario

Alice and Bob share the entangled bit \( v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \).

1. \((x, y) = (0, 0)\). Alice: \(\frac{\pi}{4}\). Bob: \(\frac{\pi}{8}\). Prob they agree:
\[
\cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
\]

2. \((x, y) = (0, 1)\). Alice: \(\frac{\pi}{4}\). Bob: \(\frac{3\pi}{8}\). Prob they agree:
\[
\cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
\]

3. \((x, y) = (1, 0)\). Alice: 0. Bob: \(\frac{\pi}{8}\). Prob they agree:
\[
\cos\left(\frac{\pi}{8} - 0\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
\]

4. \((x, y) = (1, 1)\). Alice: 0. Bob: \(3\pi/8\). Prob they agree:
\[
\cos\left(\frac{3\pi}{8} - 0\right) = \cos^2\left(\frac{3\pi}{8}\right) \sim 0.146.
\]
So prob they do not agree is \(\sim 1 - 0.146 = 0.853\).
Each Scenario

Alice and Bob share the entangled bit $\nu = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob they agree: $\cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob they agree: $\cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{8}$. Prob they agree: $\cos\left(\frac{\pi}{8} - 0\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

4. $(x, y) = (1, 1)$. Alice: 0. Bob: $\frac{3\pi}{8}$. Prob they agree: $\cos\left(\frac{3\pi}{8} - 0\right) = \cos^2\left(\frac{3\pi}{8}\right) \sim 0.146$.

So prob they do not agree is $\sim 1 - 0.146 = 0.853$.

If $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$ then the prob $a = b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$. 
Each Scenario

Alice and Bob share the entangled bit \( v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \).

1. \((x, y) = (0, 0)\). Alice: \(\frac{\pi}{4}\). Bob: \(\frac{\pi}{8}\). Prob they agree:
   \[
   \cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
   \]

2. \((x, y) = (0, 1)\). Alice: \(\frac{\pi}{4}\). Bob: \(\frac{3\pi}{8}\). Prob they agree:
   \[
   \cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
   \]

3. \((x, y) = (1, 0)\). Alice: 0. Bob: \(\frac{\pi}{8}\). Prob they agree:
   \[
   \cos\left(\frac{\pi}{8} - 0\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853.
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4. \((x, y) = (1, 1)\). Alice: 0. Bob: \(\frac{3\pi}{8}\). Prob they agree:
   \[
   \cos\left(\frac{3\pi}{8} - 0\right) = \cos^2\left(\frac{3\pi}{8}\right) \sim 0.146.
   \]
   So prob they do not agree is \(\sim 1 - 0.146 = 0.853\).

If \((x, y) \in \{(0, 0), (0, 1), (1, 0)\}\) then the prob \(a = b\) is \(\sim 0.853\), so prob of WIN is \(\sim 0.853\).

If \((x, y) = (1, 1)\) then the prob \(a \neq b\) is \(\sim 0.853\), so prob of WIN is \(\sim 0.853\).
Each Scenario

Alice and Bob share the entangled bit $\nu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob they agree:
   $\cos^2(\frac{\pi}{4} - \frac{\pi}{8}) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob they agree:
   $\cos^2(\frac{3\pi}{8} - \frac{\pi}{4}) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

3. $(x, y) = (1, 0)$. Alice: 0. Bob: $\frac{\pi}{8}$. Prob they agree:
   $\cos(\frac{\pi}{8} - 0) = \cos^2(\frac{\pi}{8}) \sim 0.853$.

4. $(x, y) = (1, 1)$. Alice: 0. Bob: $3\pi/8$. Prob they agree:
   $\cos(\frac{3\pi}{8} - 0) = \cos^2(\frac{3\pi}{8}) \sim 0.146$.
   So prob they do not agree is $\sim 1 - 0.146 = 0.853$.

If $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$ then the prob $a = b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$.
If $(x, y) = (1, 1)$ then the prob $a \neq b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$.

Hence the prob of a win is
$\sim \frac{3}{4}(0.853) + \frac{1}{4}(0.853) = 0.853$. Better than 0.8125.
Each Scenario

Alice and Bob share the entangled bit $v = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$.

1. $(x, y) = (0, 0)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{\pi}{8}$. Prob they agree:
   $\cos^2\left(\frac{\pi}{4} - \frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

2. $(x, y) = (0, 1)$. Alice: $\frac{\pi}{4}$. Bob: $\frac{3\pi}{8}$. Prob they agree:
   $\cos^2\left(\frac{3\pi}{8} - \frac{\pi}{4}\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

3. $(x, y) = (1, 0)$. Alice: $0$. Bob: $\frac{\pi}{8}$. Prob they agree:
   $\cos\left(\frac{\pi}{8} - 0\right) = \cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

4. $(x, y) = (1, 1)$. Alice: $0$. Bob: $3\pi/8$. Prob they agree:
   $\cos\left(\frac{3\pi}{8} - 0\right) = \cos^2\left(\frac{3\pi}{8}\right) \sim 0.146$.

   So prob they do not agree is $\sim 1 - 0.146 = 0.853$.

If $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$ then the prob $a = b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$.

If $(x, y) = (1, 1)$ then the prob $a \neq b$ is $\sim 0.853$, so prob of WIN is $\sim 0.853$.

Hence the prob of a win is
\[
\sim \frac{3}{4}(0.853) + \frac{1}{4}(0.853) = 0.853. \text{ Better than 0.8125.}
\]

The exact prob of winning is $\cos^2\left(\frac{\pi}{8}\right)$. 
Can Alice and Bob Do Better?

Vote Which of the following is true:

1. If Alice has then Alice and Bob have a strategy that wins the CHSH game with Prob \( \cos^2 \left( \frac{\pi}{8} \right) \) and this is known.

2. If Alice has then the max prob that Alice and Bob can achieve is \( \cos^2 \left( \frac{\pi}{8} \right) \) and this is known.

3. The question of if Alice and Bob can do better than 0.8125 is Unknown to Science.

Answer on the Next Page
Can Alice and Bob Do Better?

Vote Which of the following is true:
Can Alice and Bob Do Better?

**Vote** Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with \( \text{Prob } p > \cos^2\left(\frac{\pi}{8}\right) \) and this is known.
Can Alice and Bob Do Better?

**Vote** Which of the following is true:

1. If Alice has . . . then Alice and Bob have a strategy that wins the CHSH game with Prob $p > \cos^2\left(\frac{\pi}{8}\right)$ and this is known.
2. If Alice has . . . then the max prob that Alice and Bob can achieve is $\cos^2\left(\frac{\pi}{8}\right)$ and this is known.
Can Alice and Bob Do Better?

**Vote** Which of the following is true:

1. If Alice has ... then Alice and Bob have a strategy that wins the CHSH game with $\text{Prob } p > \cos^2\left(\frac{\pi}{8}\right)$ and this is known.

2. If Alice has ... then the max prob that Alice and Bob can achieve is $\cos^2\left(\frac{\pi}{8}\right)$ and this is known.

3. The question of if Alice and Bob can do better than 0.8125 is *Unknown to Science.*
Can Alice and Bob Do Better?

Vote Which of the following is true:

1. If Alice has . . . then Alice and Bob have a strategy that wins the CHSH game with Prob $p > \cos^2\left(\frac{\pi}{8}\right)$ and this is known.
2. If Alice has . . . then the max prob that Alice and Bob can achieve is $\cos^2\left(\frac{\pi}{8}\right)$ and this is known.
3. The question of if Alice and Bob can do better than 0.8125 is Unknown to Science.

Answer on the Next Page
Can Alice and Bob Do Better?

EVAN-I will fill this slide in once we know.
IF THEY CANNOT DO BETTER THEN ASK ABOUT IF THEY HAD MORE PAIRS OF ENTANGLED QUBITS
1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.

2. If Alice and Bob have qubits that are entangled then there is a strategy that has prob of winning $\cos^2\left(\frac{\pi}{8}\right) \sim 0.85$.

3. I am amazed that with entanglement Alice and Bob can do better.

4. I am amazed that with entanglement Alice and Bob can do so much better. I would have thought something like 0.75 + $\epsilon$. 
Final Thoughts

1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.
Final Thoughts

1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.

2. If Alice and Bob have qubits that are entangled then there is a strategy that has prob of winning \( \cos^2\left(\frac{\pi}{8}\right) \sim 0.853. \)
1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.

2. If Alice and Bob have qubits that are entangled then there is a strategy that has prob of winning $\cos^2\left(\frac{\pi}{8}\right) \sim 0.853$.

3. I am amazed that with entanglement Alice and Bob can do much better.
1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.

2. If Alice and Bob have qubits that are entangled then there is a strategy that has prob of winning \( \cos^2\left(\frac{\pi}{8}\right) \sim 0.853 \).

3. I am amazed that with entanglement Alice and Bob can do better.

4. I am amazed that with entanglement Alice and Bob can do so much better. I would have have thought something like 0.75 + \( \epsilon \).