1 Introduction

Definition 1. Let \((A, \preceq_A)\) and \((B, \preceq_B)\) be ordered sets. Then \(A, B\) are order-equivalent, denoted \(A \approx B\), if there exists an order-preserving bijection \(f : A \to B\); that is, for all \(a_1, a_2 \in A\):

\[
a_1 \preceq_A a_2 \iff f(a_1) \preceq_B f(a_2).
\]

Notation 2. Let \(a, b \in \mathbb{N}\) and \(S\) be a set.

1. \([b]\) is \(\{1, \ldots, b\}\).
2. \(\binom{S}{a}\) is the set of all \(a\)-element subsets of \(S\).
3. Let \(\text{COL} : S \to [b]\) and \(S' \subseteq S\). Then \(\text{COL}(S')\) is the codomain of \(\text{COL}\) restricted to \(S'\). Hence \(|\text{COL}(S')|\) is the number of unique elements in the codomain of \(\text{COL}\) restricted to \(S'\).

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**Definition 3.** Let $S$ be an ordered set with order type $\rho$, $S' \subseteq S$, $a, b, t \in \mathbb{N}$, and $\text{COL} : \binom{S}{a} \rightarrow [b]$ be a coloring.

1. $S'$ is homogeneous if $|\text{COL}(\binom{S'}{a})| = 1$. $S'$ is $t$-homogeneous if $|\text{COL}(\binom{S'}{a})| \leq t$.

2. Let $\text{COL} : \binom{S}{a} \rightarrow [b]$ be a coloring. $S'$ is $\rho$-$t$-homogeneous if $|\text{COL}(\binom{S'}{a})| \leq t$ and $S' \cong S$ (hence $S'$ has order type $\rho$).

**Notation.**

1. $\zeta$ is the order type of the integers. $\omega$ is the order type of the naturals. $\eta$ is the order type of the rationals.

2. Polynomials in $\omega$ are order types in the usual way.

3. For all of the order types above we use the notation given for both the order type and for the underlying set. For example, we will write things like $\text{COL} : \binom{\zeta}{a} \rightarrow [b]$.

**Definition 4.** Let $S$ be an ordered set and $\rho$ be its order type. $T(a, S)$ is the least positive integer $t$ such that, for all $b \in \mathbb{N}$, for all colorings $\text{COL} : \binom{S}{a} \rightarrow [b]$, there exists $S' \subseteq S$ such that $S'$ is $\rho$-$t$-homogeneous. Note that $t$ is independent of $b$. $T(a, S)$ is also called the Big Ramsey Degree of $\binom{S}{a}$. In other literature, for example, Zucker [8], this is sometimes written as $S \rightarrow (S)^a_{T(a,S)}$ and $S \not\rightarrow (S)^a_{T(a,S)-1}$ for all $r \in \mathbb{N}$.

This paper will be about $T(a, \zeta)$ and $T(a, \alpha)$ where $\alpha$ is an ordinal that is $< \omega^\omega$. We will not be talking about $T(a, \eta)$, however, the interested reader should know the following:

**Theorem 5.**

1. $T(2, \eta) = 2$. (This was first proven by Galvin, unpublished.)

2. For all $a$, $T(a, \eta)$ exists. (This was first proven by Laver [5].)

3. $T(a, \eta)$ is the coefficient of $x^{2a+1}$ in the Taylor series for the tangent function, hence

   $$T(a, \eta) = \frac{B_{2a+1}(-1)^{a+1}(1 - 4^{a+1})}{(2(a+1))!}$$

   where $B_{2a+1}$ is the $(2a + 1)$th Bernoulli number. (This was proven by Devlin [2], see also Vuksanovic [7] and Halpern & Lauchli [4].)

**Note 6.** The notion of $T(a, S)$ has been defined on structures other than orderings. We give an example. Let $R = (\mathbb{N}, E)$ be the Rado graph. $T(a, R)$ is the least number $t$ such that, for all $b$, for all colorings $\text{COL} : \binom{\mathbb{N}}{a} \rightarrow [b]$, there exists $H \subseteq \mathbb{N}$ where (1) $|\text{COL}(\binom{H}{a})| \leq t$, and (2) the graph induced by $H$ is isomorphic to $R$. The numbers $T(a, R)$ are known but complicated; however, $T(2, R) = 2$. See Dobrinen [3] for references and other examples.
2 Summary of Results

Ramsey’s Theorem on $\mathbb{N}$ not only gives a 1-homogenous set, it also gives a $\omega$-1-homogenous set. Hence the following is obvious.

**Theorem 7.** $T(a, \omega) = 1$.

What happens for other ordered sets? In this paper we do the following.

1. In Section 3 we show that $T(a, Z) = 2^a$. This result can be obtained by the result by Mašulović and Šobot [6] that $T(a, \omega + \omega) = 2^a$. We give a simpler and direct proof.

2. In Sections 4,5,6,8, and 9 we determine $T(a, \alpha)$ for all ordinals $\alpha < \omega^\omega$. Mašulović and Šobot [6] had shown that, for $\alpha < \omega^\omega$, $T(a, \alpha)$ is finite; however, they did not obtain the exact values of $T(a, \alpha)$.

3 $T(a, \zeta) = 2^a$

For a warmup we first prove $T(1, \zeta) = 2$ and $T(2, \zeta) = 4$.

**Theorem 8.**

1. $T(1, \zeta) = 2$.

2. $T(2, \zeta) = 4$.

**Proof.** Throughout the proof $b \in \mathbb{N}$. Think of $b$ as large.

1a) $T(1, \zeta) \leq 2$. Let $\text{COL}: \zeta \to [b]$. Let $\text{COL}': \omega \to [b]^2$ be defined by

$$\text{COL}'(x) = (\text{COL}(x), \text{COL}(-x)).$$

By Ramsey’s theorem there is a homogenous set $H'$. Clearly the set $H = H' \cup -H'$ is $\zeta$-2-homogenous.

1b) $T(1, \zeta) \geq 2$. Let $\text{COL}: \zeta \to [2]$ be the coloring that colors the positive integers and zero RED and the negative integers BLUE. Clearly there is no $\zeta$-1-homogenous set.

2a) $T(2, \zeta) \leq 4$. Let $\text{COL}: (\zeta,2) \to [b]$. Let $\text{COL}': (\zeta,2) \to [b]^4$ be defined by

$$\text{COL}'(x,y) = (\text{COL}(x,y), \text{COL}(x,-y), \text{COL}(-x,y), \text{COL}(-x,-y)).$$

By Ramsey’s theorem there is a homogenous set $H'$. Let the color of the homogenous set be $(c_1, c_2, c_3, c_4)$. Clearly the set $H = H' \cup -H'$ is $\zeta$-4-homogenous.

2b) $T(2, \zeta) \geq 4$. Let $\text{COL}: (\zeta,2) \to [4]$ be the coloring
\[
\text{COL}'(x, y) = \begin{cases} 
1 & \text{if } x, y \geq 0 \\
2 & \text{if } x \geq 0, y < 0, \text{ and } |x| \leq |y| \\
3 & \text{if } x \geq 0, y < 0, \text{ and } |x| > |y| \\
4 & \text{if } x < 0, y < 0 
\end{cases}
\]  

We leave it to the reader to show there is no \(\zeta\)-3-homogenous set. \(\square\)

**Theorem 9.** For all \(a \in \mathbb{N}\), \(T(a, \zeta) = 2^a\).

**Proof.** Throughout the proof \(b \in \mathbb{N}\). Think of \(b\) as large.

a) \(T(a, \zeta) \leq 2^a\). Let \(\text{COL}: \binom{\zeta}{a} \to [b]\). Let \(\text{COL}' : \binom{\omega}{a} \to [b]^{2^a}\) be defined by

\[
\text{COL}'(x_1, \ldots, x_a) = (\text{COL}(x_1, \ldots, x_a), \text{COL}(-x_1, x_2, x_3, \ldots, x_a), \ldots, \text{COL}(-x_1, -x_2, \ldots, -x_a)).
\]

(The output of the coloring goes through all \(2^a\) ways to negate a subset of the numbers.)

By Ramsey’s theorem there is a homogenous set \(H'\). Clearly the set \(H = H' \cup -H'\) is \(\zeta\)-2\(^a\)-homogenous.

b) \(T(a, \zeta) \geq 2^a\).

We describe a coloring \(\text{COL}: \binom{\zeta}{a} \to [2^a]\). The codomain won’t be \([2^a]\); however, it will be a set of that size.

1. Let \(\{x_1, \ldots, x_a\} \in \binom{\zeta}{a}\).
2. Let \((i_1, \ldots, i_a)\) be such that

\[|x_{i_1}| \leq |x_{i_2}| \leq \cdots \leq |x_{i_a}|.\]

We give an example that also shows how to deal with ambiguity. Assume we are given \(\{-7, 4, 0, 7\}\). We will order the absolute values as \(|0| \leq |4| \leq |-7| \leq |7|\) so we have indices \((3, 2, 1, 4)\). Note that we ordered \(|-7| \leq |7|\). This is our convention.

3. Let \(s_{i_j} = +\) if \(x_{i_j} \geq 0\) and \(-\) if \(x_{i_j} < 0\).
4. The color is \((s_{i_1}, \ldots, s_{i_a})\).

Clearly we use \(2^a\) colors. We leave it to the reader to show that there is no \(\zeta\)-(\(2^a - 1\))-homogenous set. \(\square\)
4 $T(a, \omega)$ and $T(a, \omega \cdot k)$

As noted in Theorem 7, $T(a, \omega) = 1$. In this and later sections we will look at larger ordinals. For simplicity in stating results, we will not find the big Ramsey degrees of successor ordinals such as $\omega + 1, \omega + 2, \ldots$ until Section 9

**Theorem 10.** For all $k \in \mathbb{N}$, $T(1, \omega \cdot k) = k$.

**Proof.**

**Part 1:** $T(1, \omega \cdot k) \leq k$

Let $\text{COL}: (\omega^k \upharpoonright 1) \to [b]$ for some $b$. We construct a $k$-homogeneous set $H \approx \omega \cdot k$.

We represent $\omega \cdot k$ as

$$\omega \cdot k = X_1 + \cdots + X_k$$

where each $X_i$ is a copy of $\omega$.

We can now view $\text{COL}$ as $\text{COL}: (X_1 \cup \cdots \cup X_k) \to [b]$.

For $1 \leq i \leq k$ consider the restriction of $\text{COL}$ to $(X_i \upharpoonright 1)$. Since $T(1, \omega) = 1$ by Theorem 7, there exists a 1-homogeneous $X'_i \approx X_i$. Let

$$H = X'_1 + \cdots + X'_k.$$

Then $H$ is a $k$-homogeneous set with $H \approx \omega \cdot k$.

**Part 2:** $T(1, \omega \cdot k) \geq k$

We represent $\omega \cdot k$ as

$$\omega \cdot k = X_1 + \cdots + X_k$$

where each $X_i$ is a copy of $\omega$.

We define $\text{COL}: (X_1 \cup \cdots \cup X_k) \to [k]$ by mapping all $x \in X_i$ to $i$.

We leave it to the reader to show that there is no $(k-1)$-homogeneous set $H \approx \omega \cdot k$.

**Theorem 11.** For all $a, k \in \mathbb{N}, T(a, \omega \cdot k) = k^a$.

**Proof.**

**Part 1:** $T(a, \omega \cdot k) \leq k^a$

We proceed by induction on $a$. The base case, $a = 1$, follows directly from Theorem 10. Henceforth we assume the $a-1$ case, namely that $T(a-1, \omega \cdot k) \leq k^{a-1}$.

Let $\text{COL}: (\omega^k \upharpoonright a) \to [b]$ for some finite $b$. We construct a $k^a$-homogeneous set $H \approx \omega \cdot k$.

We represent $\omega \cdot k$ as

$$\omega \cdot k = X_1 + \cdots + X_k$$

where each $X_i$ is a copy of $\omega$. We can now view $\text{COL}$ as $\text{COL}: (X_1 \cup \cdots \cup X_k) \to [b]$.

We will find $F'_1 \subseteq X_1, \ldots, F'_k \subseteq X_k$, such that the following hold.
For $1 \leq i \leq k$, $F_i' \approx X_i \approx \omega$.

$|\text{COL}(\bigcup_{a}^{F_1' \cup \cdots \cup F_k'})| \leq k^a$.

We will then set
\[
H = F_1' + \cdots + F_k'.
\]

and achieve our desired result.

**Stage 1**

For $1 \leq i \leq k$ let $F_i = \emptyset$. $F_i$ will be the set of *frozen* points from $X_i$. Such points, once *frozen*, will not be removed until Stage 2.

Throughout this stage, for $1 \leq i \leq k$:

- Some points from $X_i$ will be moved to $F_i$. Other points of $X_i$ will just be removed entirely. At the end of Stage 1, each $F_i$ will be infinite.

- We will define the coloring
\[
\text{COL}_i : F_i \mapsto \left( \binom{[b]}{k^{a-1}} \right)
\]
to be initially empty as $F_i$ starts out empty, and extend its definition as elements are added to $F_i$.

- The set $F_i$ and the coloring $\text{COL}_i$ are global variables that will be redefined throughout Stage 1.

This stage proceeds in an infinite number of rounds. In each round, we will visit each $X_i$ for $1 \leq i \leq k$. We now describe what happens in Round $r$ visiting $X_i$.

**Round $r$ Visiting $X_i$**

1. Let $y$ be the least element of $X_i$. Remove $y$ from $X_i$ and insert it into $F_i$.

2. Let $\text{COL}' : \left( \binom{X_1 \cup \cdots \cup X_k}{a-1} \right) \mapsto [b]$ be defined by
\[
\text{COL}'(\{x_1, \ldots, x_{a-1}\}) = \text{COL}(\{x_1, \ldots, x_{a-1}, y\})
\]

3. Since $T(a - 1, \omega \cdot k) \leq k^{a-1}$ by induction there exists some
\[
X_1' \subseteq X_1, \ldots, X_k' \subseteq X_k
\]
such that the following hold:

- For $1 \leq i \leq k$, $X_i' \approx X_i \approx \omega$.

- $|\text{COL}' \left( \binom{X_1' \cup \cdots \cup X_k'}{a-1} \right) | \leq k^{a-1}$.
4. For $1 \leq i \leq k$ replace $X_i$ with $X'_i$.

5. Extend the definition of $\text{COL}_i$ so that $\text{COL}_i(x) = \text{COL}'(X'_1 \cup \cdots \cup X'_k)$. Note that $\text{COL}_i(x)$ is a set of $\leq k^{a-1}$ colors.

What Do We Know After Stage 1?

Let $\{y_1, \ldots, y_a\} \in (F'_1 \cup \cdots \cup F'_k)$ with $y_1$ the element that was frozen first in Stage 1. Let $y_1$ be in $F_i$. Note that $\text{COL}(\{y_1, \ldots, y_a\}) \in \text{COL}_i(y_1)$.

Stage 2

For $1 \leq i \leq k$, since $T(1, F_i) = 1$ by Theorem 7, there exists $F'_i \subseteq F_i$ such that $|\text{COL}(F'_i)| = 1$. Let $C_i = \text{COL}_i(F'_i)$.

We will take $H = F'_1 + \cdots + F'_k$.

For all $1 \leq i \leq k$, $F'_i \approx F_i \approx \omega$, so $H \approx \omega \cdot k$.

We need to prove that

$$|\text{COL}\left(\left(\left(F'_1 \cup \cdots \cup F'_k\right)\right)\right)| \leq k^a.$$

Partition $(F'_1 \cup \cdots \cup F'_k)$ into $k$ sets $S_1, \ldots, S_k$ as follows: Let $\{y_1, \ldots, y_a\} \in (F'_1 \cup \cdots \cup F'_k)$ with $y_1$ the element that was frozen first. Let $y_1$ be in $F'_i$. Then insert $\{y_1, \ldots, y_a\}$ into $S_i$. Note that $\text{COL}(\{y_1, \ldots, y_a\}) \in C_i$ which only depends on $i$.

Hence

$$|\text{COL}\left(\left(\left(F'_1 \cup \cdots \cup F'_k\right)\right)\right)| = |\text{COL}(S_1) \cup \cdots \cup \text{COL}(S_k)| \leq |C_1| + \cdots + |C_k| \leq k \cdot k^{a-1} = k^a.$$

Part 2: $T(a, \omega \cdot k) \geq k^a$

We give a $k^a$-coloring of $(\omega^k)$ that has no $(k^a - 1)$-homogeneous $H \approx \omega \cdot k$. We represent $\omega \cdot k$ as

$$\omega \cdot k = X_1 + \cdots + X_k$$

where each $X_i \approx \omega$. We will write each $X_i$ to be a set of disjoint natural numbers. We represent an element of $\omega \cdot k$ by $(i, x)$ where the element is in $X_i$ and within $X_i$, it is the number $x$.

Before giving the coloring we give an example with $a = 5$ and $k = 200$. To color
{\( (3,12), (50,2), (110,7), (110,7777), (117,3) \)}

we do the following:

1. Order the ordered pairs by their second coordinates. So we have
   \( ((50,2), (117,3), (110,7), (3,12), (110,7777)) \).

   (Since all of the \( X_i \)'s are disjoint the second coordinates are all different, so there will never be a tie.)

2. The color is the sequence of first coordinates. So the color is
   \( (50,117,110,3,110) \).

Notice that the number of colors is the number of 5-tuple where each number is in \( \{1, \ldots, 200\} \). Hence there are \( 200^5 \) colors.

In general, given \( \{(i_1,x_1), \ldots, (i_a,x_a)\} \):

1. Order the ordered pairs by the second coordinate.

2. The color is the sequence of first coordinates.

Notice that the number of colors is the number of \( a \)-tuples where each number is in \( \{1, \ldots, k\} \). Hence there are \( k^a \) colors.

We leave it to the reader to show that there can be no \( (k^a - 1) \)-homogeneous \( H \cong \omega \cdot k \).

\[ \square \]

5 \( T(2, \omega^2) \)

**Theorem 12.** \( T(2, \omega^2) = 4 \).

**Proof.**

Every element of \( \omega^2 \) is a linear expression in \( \omega \) with non-negative integer coefficients. For example, \( \omega \cdot 17 + 8 \in \omega^2 \).

**Part 1: \( T(2, \omega^2) \leq 4 \)**

Let

\[ \text{COL}: \binom{\omega^2}{2} \to [b] \]

be an arbitrary coloring of \( \binom{\omega^2}{2} \) for some finite \( b \). We define four colorings \( f_1, f_2, f_3, f_4 \) from domain \( \binom{\omega}{4} \) to codomain \( \binom{\omega^2}{2} \) and then use them to define a coloring from \( \binom{\omega}{4} \) to \([b] \times [b] \times [b] \times [b] \). In what follows, we assume \( x_1 < x_2 < x_3 < x_4 \).
\[ f_1: \binom{\omega}{4} \to \binom{\omega^2}{2} \text{ is defined by} \\
(x_1, x_2, x_3, x_4) \mapsto \text{COL}\{\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4\}. \]

\[ f_2: \binom{\omega}{4} \to \binom{\omega^2}{2} \text{ is defined by} \\
(x_1, x_2, x_3, x_4) \mapsto \text{COL}\{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_4\}. \]

\[ f_3: \binom{\omega}{4} \to \binom{\omega^2}{2} \text{ is defined by} \\
(x_1, x_2, x_3, x_4) \mapsto \text{COL}\{\omega \cdot x_1 + x_4, \omega \cdot x_2 + x_3\}. \]

\[ f_4: \binom{\omega}{4} \to \binom{\omega^2}{2} \text{ is defined by} \\
(x_1, x_2, x_3, x_4) \mapsto \text{COL}\{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\}. \]

\[ \text{COL'}: \binom{\omega}{4} \to [b] \times [b] \times [b] \times [b] \text{ is defined by} \\
(x_1, x_2, x_3, x_4) \mapsto (f_1(x_1, x_2, x_3, x_4), f_2(x_1, x_2, x_3, x_4), f_3(x_1, x_2, x_3, x_4), f_4(x_1, x_2, x_3, x_4)). \]

Apply Theorem 7 on \( \text{COL'} \) to find some \( N \approx \omega \) where \( |\text{COL'}(\binom{N}{4})| = 1 \). Enumerate \( N = \{x_1, x_2, \ldots\} \) with \( x_1 < x_2 < \cdots \). Let

\[ H = \omega \cdot x_1 + x_2, \omega \cdot x_1 + x_6, \omega \cdot x_1 + x_{10}, \ldots, (\omega \text{ times}) \]
\[ \omega \cdot x_3 + x_4, \omega \cdot x_3 + x_{12}, \omega \cdot x_3 + x_{20}, \ldots, (\omega \text{ times}) \]
\[ \omega \cdot x_5 + x_8, \omega \cdot x_5 + x_{24}, \omega \cdot x_5 + x_{40}, \ldots, (\omega \text{ times}) \]
\[ \vdots \ (\omega \text{ times}) \]

Then \( H \approx \omega^2 \). For any edge \( \{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\} \in \binom{H}{2} \) with \( \omega \cdot y_1 + y_2 < \omega \cdot y_3 + y_4 \), either \( y_1 \neq y_3 \) or \( y_1 = y_3 \).

- If \( y_1 \neq y_3 \), then \( y_1 < y_3 \) by the ordering of the two elements and \( y_2 \neq y_4 \) by the construction of \( H \). We also have \( y_1 < y_2, y_1 < y_4, \) and \( y_3 < y_4 \) by the construction of \( H \). Then either \( y_1 < y_2 < y_3 < y_4, y_1 < y_3 < y_2 < y_4, \) or \( y_1 < y_3 < y_4 < y_2 \). In each of the three cases, because we have \( f_1(y_1, y_2, y_3, y_4) \in Y, f_2(y_1, y_3, y_2, y_4) \in Y, \) and \( f_3(y_1, y_3, y_4, y_2) \in Y \), we know \( \text{COL}\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\} \in Y \).

- If \( y_1 = y_3 \), then \( y_2 < y_4 \) by the ordering of the elements and so \( y_1 = y_3 < y_2 < y_4 \) by the construction of \( H \). Because \( f_4(y_1, y_2, y_4, 1) \in Y \), we know \( \text{COL}\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\} \in Y \).
In all cases, \( \text{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y \) so 
\[
\text{COL} \left( \binom{H}{2} \right)
\]
expresses at most \(|Y| = 4\) colors. Because \( \text{COL} \) was arbitrary, \( T(2, \omega^2) \leq 4 \).

**Part 2:** \( T(2, \omega^2) \geq 4 \) Let

\[
\text{COL}: \left( \binom{\omega^2}{2} \right) \rightarrow [4]
\]

\[
\{\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4\} \mapsto \begin{cases} 
1 & x_1 < x_2 < x_3 < x_4 \\
2 & x_1 < x_3 < x_2 < x_4 \\
3 & x_1 < x_3 < x_4 < x_2 \\
4 & x_1 = x_3 < x_2 < x_4 \text{ or otherwise}
\end{cases}
\]

where \( \omega \cdot x_1 + x_2 < \omega \cdot x_3 + x_4 \). Color 4 could be formatted as simply “otherwise”, but the specific part that makes color 4 present in any order-equivalent subset is the \( x_1 = x_3 < x_2 < x_4 \) (the “otherwise” can be filtered out). We leave it to the reader to show that there exists no 3-homogeneous order-equivalent subset.

\[
\square
\]

### 6 Strong Colorings

We will use a concept called **Strong Colorings** to prove general results about \( T(a, \omega^d) \) and beyond. The concept behind strong colorings is built on the ideas of Blass et al. [1]. We motivate the concept by looking at the proof of Theorem 12.

The proof of Theorem 12 used four functions \( f_1, f_2, f_3, f_4 \). These functions were specifically chosen to cover \( H \) in a way where the color of every edge in \( H \) was in the output of some \( f_1, f_2, f_3, f_4 \). We note a function that was not used:

\[
f: \binom{\omega^4}{4} \rightarrow \binom{\omega^2}{2}
\]

is defined by

\[
(x_1, x_2, x_3, x_4) \mapsto \text{COL}(\{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_3\}).
\]

We didn’t use \( f \) in the lower bound proof because \( f \) didn’t cover any edges in \( H \): we built \( H \) in a way where distinct copies of \( \omega \) had distinct finite coefficients. Since \( x_1 \neq x_2 \), we know that the elements \( \omega \cdot x_1 + x_3 \) and \( \omega \cdot x_2 + x_3 \) couldn’t both be from \( H \) no matter the values of \( x_1, x_2, \) and \( x_3 \).

We could have designed \( H \) differently to require more than 4 functions to cover it, but that would have weakened the result of Theorem 12. We will define a notion of colorings that \( f_1, f_2, f_3, f_4 \) qualify but \( f \) does not. In later sections, we will show how to count these colorings, and how these colorings are linked to Big Ramsey degrees.
**Definition 13.** For integers $a, d, k \geq 0$, we say there are $a$ elements in each $e \in \left(\omega^d \cdot k\right)_a$. For $1 \leq q \leq a$, we denote each element as

$$\omega^d \cdot y_q + \omega^{d-1} \cdot x_{q,d-1} + \omega^{d-2} \cdot x_{q,d-2} + \cdots + \omega \cdot x_{q,1} + x_{q,0}.$$ 

This leads to $a y_q$ variables in an $e \in \left(\omega^d \cdot k\right)_a$, each an integer ranging from $0 \leq y_q < d$. We also have $a \cdot d x_{qn}$ variables, with $1 \leq q \leq a$ and $0 \leq n < d$. Note that each $x_{qn}$ can be any finite nonnegative integer.

A **strong coloring** is first defined as a selection of $y$ variables $Y \in \{0, 1, \ldots, k - 1\}^a$. Each $y_q$ is set to the $q$th element of $Y$. Then, the $x_{qn}$ variables in $\left(\omega^d \cdot k\right)_a$ with $< \text{ or } =$ signs between them are permuted in a way that satisfies the below criteria.

1. If $d \geq 1$, $x_{i0} < x_{j0}$ for all $i < j$ (the element indices are ordered by their lowest-dimension variable). If $d = 0$, $y_i < y_j$ for all $i < j$.

2. $y_i \neq y_j \rightarrow x_{in} \neq x_{jn}$ for all $0 \leq n < d$ (Elements that have a different $y$ value have all different $x$ values).

3. $x_{qa} < x_{qb}$ for all $a > b$ (the high-dimension variables of each element are strictly less than the low-dimension variables).

4. $x_{ma} = x_{nb} \rightarrow a = b$ (only variables with the same dimension can be equal).

5. $x_{in} \neq x_{jn} \rightarrow x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$ and $i \neq j$ (elements that differ in a high-dimension variable differ in all lower-dimension variables).

An example of a strong coloring for the expression $\left(\omega^2 \cdot 2\right)_a$ would be

$$Y = (0, 0), \ x_{11} = x_{21} < x_{10} < x_{20}.$$ 

Note that there is only one possible value of $Y$ because $k = 1$ in the example. We say that two strong colorings are equivalent if and only if their $Y$ tuples are the same and they are logically equivalent; that is, identical up to permutation of variables within equivalence classes.

**Definition 14.** The size of a strong coloring as how many equivalence classes its $x$ variables form: for example, $x_{11} = x_{21} < x_{10} < x_{20}$ would have size $p = 3$ regardless of $y$ variables. Clearly a strong coloring’s size $p$ can be no larger than $a \cdot d$, how many $x$ variables $\left(\omega^d \cdot k\right)_a$ has.

**Definition 15.**

1. $P_p \left(a, \omega^d \cdot k\right)$ is the number of strong colorings with size $p$ there are for $\left(\omega^d \cdot k\right)_a$.

2. $P \left(a, \omega^d \cdot k\right)$ is the number of strong colorings there are for $\left(\omega^d \cdot k\right)_a$ regardless of size.

**Definition 16.** We say that an edge satisfies a strong coloring if its $y_q$ variables match $Y$ and the $x_{qn}$ variables match the ordering of the strong coloring. Note that some edges might not satisfy any strong colorings.
Section 7 \( T(a, \omega^d) = P(a, \omega^d) \)

This section is devoted to the case where \( k = 1 \). When defining a strong coloring when \( k = 1 \), \( Y \in \{0\}^d \) is locked to the single value \((0,0,\ldots,0)\) so each \( y_q = 0 \). Then all \( y_q \) values are the same, so criterion 2 of Definition 13 is always satisfied. In this section, our proofs will focus on only the permutation of the \( x_{qn} \) variables and the remaining criteria.

This section’s aim is to show equality between Big Ramsey degrees and strong colorings. To motivate this, we start with a recurrence for strong colorings.

**Lemma 17.** For integers \( a, d \geq 0 \),

\[
P_p(a, \omega^d) = \begin{cases} 
0 & d = 0 \land a \geq 2 \\
1 & a = 0 \land p = 0 \\
0 & a = 0 \land p \geq 0 \\
1 & d = 0 \land a = 1 \land p = 0 \\
0 & d = 0 \land a = 1 \land p \geq 1 \\
1 & d = 1 \land a \geq 1 \land a = p \\
0 & d = 1 \land a \geq 1 \land a \neq p \\
\sum_{j=1}^{a} \sum_{i=0}^{p-1} (p-1)_i P_i(j, \omega^{d-1}) P_{p-i}(a-j, \omega^d) & d \geq 2 \land a \geq 1 
\end{cases}
\]

**Proof.** First consider when \( a \geq 2 \) and \( d = 0 \). We need \( Y \in \{0\}^a \) so \( Y = (0,0,\ldots,0) \) so \( y_1, y_2 = 0 \). But by criterion 1 of Definition 13, since \( d = 0 \) we need \( y_1 < y_2 \), so no strong colorings are possible, regardless of size \( p \). This aligns with the first case of the result.

Consider when \( a = 0 \): now criterion 1 is vacuously satisfied because \( Y = \emptyset \). Since \( a \cdot d = 0 \), there are no \( x \) variables to permute. Therefore there is only one strong coloring, and it has size \( p = 1 \). This aligns with the second and third cases of the result.

When both \( d = 0 \) and \( a \leq 1 \), criterion 1 of Definition 13 can be vacuously satisfied with \( Y \in \{0\}^a \) because there are either zero or one \( y_q \) variables. Again, because \( a \cdot d = 0 \), there are no \( x \) variables to permute so there is only one empty strong coloring with size \( p = 0 \), aligning with the fourth and fifth cases of the result.

Now consider \( a \geq 1 \), \( d = 1 \). To ensure criterion 1 of Definition 13, each of the \( a x_{q,0} \) variables can only form one strong coloring \( x_{1,0} < x_{2,0} < \ldots < x_{a,0} \) with size \( a \) so \( P_a(a, \omega^d) = 1 \) and \( P_p(a, \omega^d) = 0 \) for \( p \neq a \). This aligns with the sixth and seventh cases of the result.

Finally, consider \( a \geq 1 \), \( d \geq 2 \). We will prove the final case of our result by showing the process for combining strong colorings described below creates all possible strong colorings of an expression.

For arbitrary integers \( a \geq 1 \), \( d \geq 2 \), and \( p \geq 0 \), let \( 1 \leq j \leq a \) and \( 0 \leq i \leq p - 1 \) be integers. As we proceed through the process, we will work with an example of \( a = 4 \), \( d = 5 \), \( p = 13 \), \( j = 2 \), and \( i = 5 \).
We will create
\[ \binom{p-1}{i} P_i \left( j, \omega^{d-1} \right) P_{p-1-i} \left( a-j, \omega^d \right) \]
strong colorings, with each strong coloring having \( j \) elements equal in their highest-dimension variable with those \( j \) elements having a combined size (i.e. count of distinct variables) of \( i \).

Let \( \tau_1 \) represent one of the \( P_i \left( j, \omega^{d-1} \right) \) strong colorings of \( \binom{\omega^{d-1}}{j} \) with size \( i \), and \( \tau_2 \) represent one of the \( P_{p-1-i} \left( a-j, \omega^d \right) \) strong colorings of \( \binom{\omega^d}{a-j} \) with size \( p-1-i \). In our example, let
\[
\tau_1: x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{1,1} = x_{2,1} < x_{1,0} < x_{2,0} \\
\tau_2: x_{1,4} = x_{2,4} < x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{1,1} < x_{1,0} < x_{2,0}.
\]

Then we can combine each \( \tau_1 \) and \( \tau_2 \) to form \( \binom{p-1}{i} \) unique new strong colorings of size \( p \): Reindex each variable \( x_{q,n} \) of \( \tau_2 \) to \( x_{q+j,n} \), and permute the equivalence classes of the strong colorings together, preserving each strong coloring’s original ordering of its own equivalence classes: there are \( \binom{p-i}{j} \) ways to do this. In our example, after reindexing we have
\[
\tau_1: x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{1,1} = x_{2,1} < x_{1,0} < x_{2,0} \\
\tau_2: x_{3,4} = x_{4,4} < x_{3,3} = x_{4,3} < x_{3,2} = x_{4,2} < x_{3,1} < x_{3,0} < x_{4,0}
\]
and one of the \( \binom{12}{5} \) permutations is
\[
x_{3,4} = x_{4,4} < x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{3,3} = x_{4,3} < x_{3,2} = x_{4,2} < x_{1,1} = x_{2,1} < x_{4,1} < x_{1,0} < x_{3,1} < x_{3,0} < x_{2,0} < x_{4,0}.
\]
This new strong coloring likely breaks criterion 1 of Definition 13; for each \( 1 \leq q \leq a \), reindex each \( x_{q,n} \) according to where \( x_{q,0} \) is in the ordering of all \( x_{i,0} \). In our example, we have \( x_{1,0} < x_{3,0} < x_{2,0} < x_{4,0} \); after swapping indices 2 and 3 to enforce criterion 1 we have
\[
x_{2,4} = x_{4,4} < x_{1,3} = x_{3,3} < x_{1,2} = x_{2,2} < x_{3,2} = x_{4,3} < x_{2,1} = x_{4,2} < x_{1,1} = x_{3,1} < x_{4,1} < x_{1,0} < x_{2,1} < x_{2,0} < x_{3,0} < x_{4,0}.
\]
There are now \( d \cdot (a-j) + (d-1) \cdot j = a \cdot d - j \) variables in the strong coloring. There are exactly \( j \) variables of the form \( x_{q,d-1} \) for \( 1 \leq i \leq j \) that are not in the strong coloring; insert one equivalence class \( x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1} \) at the front of the new strong coloring, bringing its size to \( p \). We add \( x_{1,4} = x_{3,4} \) in our example to get
\[
x_{1,4} = x_{3,4} < x_{2,4} = x_{4,4} < x_{1,3} = x_{3,3} < x_{1,2} = x_{3,2} < x_{2,2} = x_{4,3} < x_{2,1} = x_{4,2} < x_{1,1} = x_{3,1} < x_{4,1} < x_{1,0} < x_{2,1} < x_{2,0} < x_{3,0} < x_{4,0}.
\]
Each strong coloring is unique by the \( \tau_1 \) and \( \tau_2 \) used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into \( \tau_1 \) and \( \tau_2 \) by whether their indices were in the leading equivalence class and reindexing.
Therefore the final case of the result holds.

We claim each strong coloring created by this process has the properties described by Definition 13: Because the high-dimension equivalence class \( x_{q_1, d-1} = x_{q_2, d-1} = \cdots = x_{q_j, d-1} \) was added at the start of the strong coloring, the high-dimension coefficients of each term are smaller than the low-dimension coefficients. Criterion 1 is satisfied by reindexing the variables. The remaining criteria are satisfied because \( \tau_1 \) and \( \tau_2 \) satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount strong colorings.

We also claim that every strong coloring of \( \binom{\omega^d}{a} \) is counted by this process: each can be mapped to some \( \tau_1 \) and \( \tau_2 \) that create it by a similar argument to proving that the process creates unique strong colorings. Every strong coloring of \( \binom{\omega^d}{a} \) must have a leading equivalence class of \( x_{q_1, d-1} = x_{q_2, d-1} = \cdots = x_{q_j, d-1} \) to satisfy Definition 13 (the equivalence class might only contain one variable); taking only the variables \( x_{q,n} \) with indices appearing in that equivalence class (but not those variables in the equivalence class itself) forms \( \tau_1 \), a strong coloring for \( \binom{\omega^{d-1}}{a-1} \). The variables with \( q \) indices not in the equivalence class form \( \tau_2 \), a strong coloring for \( \binom{\omega^d}{a-j} \). The original strong coloring of \( \binom{\omega^d}{a} \) is counted by interleaving \( \tau_1 \) with \( \tau_2 \) and inserting the leading equivalence class of \( x_{q_1, d-1} = x_{q_2, d-1} = \cdots = x_{q_j, d-1} \). Therefore the final case of the result holds.

\[ \text{7.1 \quad } T(a, \omega^d) \geq P(a, \omega^d) \]

**Theorem 18.** For integers \( a, d \geq 0 \), \( T(a, \omega^d) \geq P(a, \omega^d) \).

**Proof.** If \( P(a, \omega^d) = 0 \), this is satisfied vacuously because \( T(a, \omega^d) \geq 0 \). Now we can assume \( P(a, \omega^d) \geq 1 \). Let \( E = \binom{\omega^d}{a} \). Note that all strong colorings of \( E \) are disjoint from each other. That is, for any edge \( e \in E \), if \( e \) satisfies \( \tau' \), then it does not satisfy any nonequivalent strong coloring of \( E \). This is because if \( e \) were to satisfy two strong colorings \( \tau_1 \) and \( \tau_2 \), then \( \tau_1 \) and \( \tau_2 \) must share the same equivalence classes and order, so the strong colorings must be equivalent. Therefore, we can index them \( \tau_1 \ldots \tau_{P(a,\omega^d)} \) and construct a coloring

\[
\text{COL: } E \to [P(a, \omega^d)]
\]

\[
e \mapsto \begin{cases} 
 1 & e \text{ satisfies } \tau_i \\
 1 & \text{otherwise}
\end{cases}
\]

Similar to Theorem 12, our coloring has two ways to output color 1, both through satisfaction of \( \tau_1 \) and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of \( \tau_1 \).
We will show that there is no $\omega^d \cdot (P(a, \omega^d) - 1)$-homogeneous set: for all $H \approx \omega^d$ and for every strong coloring $\tau$ of $E$, there exists some $e \in \binom{H}{a}$ that satisfies $\tau$.

For arbitrary $H \approx \omega^d$ and $\tau$, we will find $z_{qn}$ where

$$\{\omega^{d-1}z_{1,d-1} + \cdots + \omega^1z_{1,1} + z_{1,0}, \ldots, \omega^{d-1}z_{n,d-1} + \cdots + \omega^1z_{a,1} + z_{a,0}\}$$

satisfies $\tau$.

We will do this by assigning values to each $z_{qn}$ according to where the equivalence class containing $x_{qn}$ is found in $\tau$, moving left to right. By criterion 3 of Definition 13, each $z_{qn}$ will be assigned before $z_{q,n-1}$. As we do this, we will ensure that if the leftmost unassigned value in $\tau$ is $z_{qn}$, then

$$\{\omega^{d-1}z_{q,d-1} + \cdots + \omega^{n+1}_{q,n+1} + \omega^nc_n + \omega^{n-1}_{c_{n-1}} + \cdots + \omega^1c_1 + c_0 \mid c_i \in \mathbb{N}\} \cap H \approx \omega^{n+1}.$$

By criterion 3 of Definition 13, the leftmost variable in $\tau$ must be $x_{q,d-1}$. Before any values are assigned, it is clear that

$$\{\omega^{d-1}c_{d-1} + \cdots + \omega^1c_1 + c_0 \mid c_i \in \mathbb{N}\} = \omega^d$$

and because $H \subseteq \omega^d$, $\omega^d \cap H = H \approx \omega^d$.

We know by criterion 4 of Definition 13 that all variables in an equivalence class must have the same dimension $d$. Let the leftmost equivalence class in $\tau$ be $x_{q_1,n} = x_{q_2,n} = \cdots = x_{q_m,n}$. By criterion 3 we know that each $x_{q_i,\ell}$ for $1 \leq i \leq m$ and $\ell > n$ appeared to the left of this equivalence class and has already been assigned a value, and by criterion 5 we know the values for each dimension are the equal: for all $\ell > n$ and $1 \leq i \leq m$, $z_{q_i,\ell} = z_{q_1,\ell}$.

By our previous steps, we have that

$$\{\omega^{d-1}z_{q_1,d-1} + \cdots + \omega^{n+1}_{q_1,n+1} + \omega^nc_n + \omega^{n-1}_{c_{n-1}} + \cdots + \omega^1c_1 + c_0 \mid c_i \in \mathbb{N}\} \cap H \approx \omega^{n+1}.$$

Then there exist some value $z'$ where

$$\{\omega^{d-1}z_{q_1,d-1} + \cdots + \omega^{n+1}_{q_1,n+1} + \omega^dz' + \omega^{n-1}_{c_{n-1}} + \cdots + \omega^1c_1 + c_0 \mid c_i \in \mathbb{N}\} \cap H \approx \omega^n$$

where $z'$ is greater than all previously assigned values. Then for $1 \leq i \leq m$, assign $z_{q_i,n}$ to be $z'$.

We can repeat this process to find $z_{qn}$ that satisfy every strong coloring of $E$ for arbitrary $H \approx \omega^d$. Therefore for all $H \approx \omega^d$, $|\text{COL}(\binom{\tilde{S}}{a})| \geq P(a, \omega^d)$ so $T(a, \omega^d) \geq P(a, \omega^d)$. \qed
To show that strong colorings bound Big Ramsey degrees above, we will require the lemma below.

**Lemma 19.** For integers \( a, d \geq 0 \) and \( N \approx \omega \), there exists some \( H \approx \omega^d \) where for all \( e \in \binom{H}{a} \), \( e \) satisfies a strong coloring of \( \binom{\omega^d}{a} \) and each coefficient in \( e \) is contained in \( N \).

**Proof.** Because \( N \approx \omega \), we can index it \( x_1, x_2, x_3, \ldots \) with \( x_1 < x_2 < x_3 < \cdots \). We proceed by induction on \( d \). When \( d = 0 \), \( \omega^0 \approx 1 \) and so \( H = \{ x_1 \} \) suffices. When \( d = 1 \), \( \omega^1 \approx \omega \) so \( H = N \) suffices.

For \( d \geq 2 \), partition \( N \) into infinite sets order-equivalent to \( \omega \) as follows:

\[
X_0 = \{ x_1, x_3, x_5, \ldots \} \\
X_1 = \{ x_2, x_6, 10, \ldots \} \\
X_2 = \{ x_4, x_{12}, x_{20}, \ldots \} \\
X_3 = \{ x_8, x_{24}, x_{40}, \ldots \} \\
\vdots
\]

Now apply the inductive hypothesis on each \( X_i \) with \( i \geq 1 \), yielding \( S_i \approx \omega^{d-1} \) for all \( i \geq 1 \). Then for all \( i \geq 1 \), for all \( e \in \binom{S_i}{a} \), \( e \) satisfies a strong coloring of \( \binom{\omega^{d-1}}{a} \) and each coefficient of \( e \) is contained in \( X_i \). For all \( i \), let \( S_i = \{ y_{i,1}, y_{i,2}, \ldots \} \). Then let

\[
H = \omega^{d-1} x_1 + y_{1,1}, \omega^{d-1} x_1 + y_{1,2}, \omega^{d-1} x_1 + y_{1,3}, \ldots, (\omega^{d-1} \text{ times}) \\
\omega^{d-1} x_3 + y_{2,1}, \omega^{d-1} x_3 + y_{2,2}, \omega^{d-1} x_3 + y_{2,3}, \ldots, (\omega^{d-1} \text{ times}) \\
\omega^{d-1} x_5 + y_{3,1}, \omega^{d-1} x_5 + y_{3,2}, \omega^{d-1} x_5 + y_{3,3}, \ldots, (\omega^{d-1} \text{ times}) \\
\vdots \quad (\omega \text{ times})
\]

Then we have \( H \approx \omega^d \). For any edge \( e \in \binom{S}{a} \), index its variables to satisfy criterion 1 of Definition 13. Then criterion 3 is satisfied inductively for variables with dimensions lower than \( d - 1 \). Because \( \min X_i = x_{2i} \) for all \( i \) and \( 2i - 1 < 2^d \) for all integers \( i \geq 1 \), we have \( x_{2i-1} < x \) for all \( x \in X_i \) so criterion 3 is satisfied by \( e \). Because \( X_0 \) is disjoint with all \( X_i \) with \( i \geq 1 \), criterion 4 is satisfied for variables with dimension \( d - 1 \) and by induction, it’s satisfied for lower dimensions. Because \( X_i \) is disjoint with \( X_j \) for all \( i \neq j \), we know that elements that differ in variables with dimension \( d - 1 \) differ in all lower-dimension variables. The induction with the previous statement satisfies criterion 5. Therefore \( e \) satisfies a strong coloring of \( \binom{\omega^d}{a} \). \( \square \)

**Theorem 20.** For integers \( a, d \geq 0 \), \( T(a, \omega^d) \leq P(a, \omega^d) \).

**Proof.** Let \( E = \binom{\omega^d}{a} \) and

\[
\text{COL}: E \to [b]
\]
be an arbitrary coloring of $E$ for some finite $b$.

Enumerate the strong colorings of $E$ from $\tau_1$ to $\tau_{\mathcal{P}(a,\omega^d)}$. We know the maximum size of any strong coloring of $E$ is $a \cdot d$. For each $\tau_i$, let

$$f_i: \left(\frac{\omega}{a \cdot d}\right) \to E$$

where if $\tau_i$ has size $p$, $f_i$ maps $X$ to the unique $e \in E$ where $e$ satisfies $\tau_i$ and the $p$ equivalence classes of $e$ are made up of the $p$ least elements of $X$. For example, one strong coloring of $\left(\frac{\omega^2}{2}\right)$ is

$$x_{11} = x_{21} < x_{10} < x_{20}.$$ 

The corresponding $f_i$ would be

$$f_i: \left(\frac{\omega}{4}\right) \to \left(\frac{\omega^2}{2}\right)$$

$$(x_1, x_2, x_3, x_4) \mapsto \text{COL}(\{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that $x_4$ is “wasted” by $f_i$—this is because the example strong coloring has size 3, but the maximum sized strong coloring for $\left(\frac{\omega^2}{2}\right)$ has size 4.

Then, define

$$\text{COL}'\left(\frac{\omega}{a \cdot d}\right) \to [b] \times [b] \times \cdots \times [b] \quad \quad \text{P(a,}\omega^d\text{) times}$$

$$X \mapsto (f_1(X), f_2(X), \ldots, f_{\mathcal{P}(a,\omega^d)}(X))$$

and apply Theorem 7 to find some $N \approx \omega$ where

$$\text{COL}'\left(\left(\frac{N}{a \cdot d}\right)\right)$$

expresses only one tuple $Y$ containing $|Y| = \mathcal{P}(a,\omega^d)$ colors.

Apply Lemma 19 to find some $H \approx \omega^d$ with the properties listed in Lemma 19. Now we claim

$$\text{COL}\left(\left(\frac{H}{a}\right)\right)$$

expresses at most $\mathcal{P}(a,\omega^d)$ colors.

By Lemma 19, each element $e \in \binom{H}{a}$ satisfies a strong coloring of $E$. Then for some arbitrary edge $e$, let $e$ satisfy $\tau_i$ with size $p \leq a \cdot d$. Then take the $p$ unique values in $e$, and if necessary, insert any new larger natural values from $N$ to form a set of $a \cdot d$ values; denote this $X \in \binom{N}{a \cdot d}$. We know $\text{COL}'(X) = Y$ so by the definition of $\text{COL}'$, $\text{COL}(e) \in Y$. Because $|Y| = \mathcal{P}(a,\omega^d)$, $T(a,\omega^d) \leq \mathcal{P}(a,\omega^d)$. 

\[ \square \]
7.3 \( T(a, \omega^d) = P(a, \omega^d) \)

**Theorem 21.** For all \( a, d \in \mathbb{N} \), \( T(a, \omega^d) = P(a, \omega^d) \).

**Proof.** By Theorem 18, \( T(a, \omega^d) \geq P(a, \omega^d) \). By Theorem 20, \( T(a, \omega^d) \leq P(a, \omega^d) \). The result follows.

8 \( T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k) \)

We will now use the theory we developed for the case \( k = 1 \) to prove results for arbitrary \( k \). We’ll first extend the recurrence from Lemma 17.

**Lemma 22.** For integers \( a, d, k \geq 0 \),

\[
P_p (a, \omega^d \cdot k) = \begin{cases} 
0 & \text{if } d = 0 \wedge a > k \\
1 & \text{if } a = 0 \wedge p = 0 \\
0 & \text{if } a = 0 \wedge p \geq 0 \\
\binom{k}{a} & \text{if } d = 0 \wedge 1 \leq a \leq k \wedge p = 0 \\
0 & \text{if } d = 0 \wedge 1 \leq a \leq k \wedge p \geq 1 \\
k^a & \text{if } d = 1 \wedge a \geq 1 \wedge a = p \\
0 & \text{if } d = 1 \wedge a \geq 1 \wedge a \neq p \\
\sum_{j=1}^{a} \sum_{i=0}^{p-1} \binom{p-1}{i} P_i (j, \omega^{d-1}) P_{p-1-i} (a-j, \omega^d \cdot k) & \text{if } d \geq 2 \wedge a \geq 1
\end{cases}
\]

**Proof.** First consider when \( a > k \) and \( d = 0 \). We need \( Y \in \{0, 1, \ldots, k-1\}^a \) so at most \( k \) unique values in \( Y \) are possible. But by criterion 1 of Definition 13, since \( d = 0 \) we need a unique values in \( Y \), so no strong colorings are possible, regardless of size \( p \). This aligns with the first case of the result.

Consider when \( a = 0 \): now criterion 1 is vacuously satisfied because \( Y = \emptyset \). Since \( a \cdot d = 0 \), there are no \( x \) variables to permute. Therefore there is only one strong coloring, and it has size \( p = 1 \). This aligns with the second and third cases of the result.

When both \( d = 0 \) and \( a \leq k \), criterion 1 of Definition 13 can be satisfied with \( Y \) being any permutation of \( a \) integers, with \( k \) possibilities. This leads to \( \binom{k}{a} \) feasible combinations. Again, because \( a \cdot d = 0 \), there are no \( x \) variables to permute so there are \( \binom{k}{a} \) empty strong colorings with size \( p = 0 \), aligning with the fourth and fifth cases of the result.

Now consider \( a \geq 1, d = 1 \). To ensure criteria 1 of Definition 13, each of the \( a \) \( x_{q,0} \) variables can only form one permutation \( x_{1,0} < x_{2,0} < \ldots < x_{a,0} \) with size \( a \). Because all \( x \) values are distinct and \( d = 1 \), \( Y \) is not restricted by any criteria and can be any of the \( k^a \) elements in \( \{1, 2, \ldots, k-1\}^a \). Therefore \( P_a (a, \omega^d) = k^a \) and \( P_p (a, \omega^d) = 0 \) for \( p \neq a \). This aligns with the sixth and seventh cases of the result.
Finally, consider $a \geq 1, d \geq 2$. We will prove the final case of our result by showing the process for combining strong colorings described below creates all possible strong colorings of an expression.

For arbitrary integers $a \geq 1, d \geq 2, k \geq 0$, and $p \geq 0$, let $1 \leq j \leq a$ and $0 \leq i \leq p - 1$ be integers.

We will create

$$k\binom{p-1}{i} P_i (j, \omega^{d-1}) P_{p-1-i} (a-j, \omega^d \cdot k)$$

strong colorings, with each strong coloring having $j$ elements equal in their highest-dimension variable with those $j$ elements having a combined size (i.e. count of distinct variables) of $i$.

Let $\tau_1$ represent one of the $P_i (j, \omega^{d-1})$ strong colorings of $(\omega^{d-1}_j)$ with size $i$, and $\tau_2$ represent one of the $P_{p-1-i} (a-j, \omega^d)$ strong colorings of $(\omega^d_{a-j})$ with size $p - 1 - i$.

Then we can combine each $\tau_1$ and $\tau_2$ to form $k\binom{p-1}{i}$ unique new strong colorings of size $p$: Reindex $\tau_2$, permute the equivalence classes, and insert a leading equivalence class $x_{q_1, d-1} = x_{q_2, d-1} = \cdots = x_{q_j, d-1}$ as in the proof of Lemma 17. This leads to $k\binom{p-1}{i}$ new permutations of the $x$ variables.

Because $\tau_1$ was a strong coloring for $(\omega^{d-1}_j)$, each of its $y$ values were 0. Now that we are creating a strong coloring for $(\omega^d_{a-j})$, we can choose the $y$ coefficients to be composed of values between 0 and $k - 1$. By criterion 2 of Definition 13, because all elements from $\tau_1$ are bound together in a leading high-dimension equivalence class, they must all have equal $y$ values. This leads to $k$ options for these $y$ values; with the options of permuting the $x$ variables, $k\binom{p-1}{i}$ ways to create a new strong coloring.

For the new strong coloring’s $Y$ tuple, we assign each element originally from $\tau_2$ with its original $y$ value (likely at a different index due to reindexing). Then, the remaining elements from $\tau_1$ are given all the same $y$ value from one of the $k$ options.

Each strong coloring is unique by the $\tau_1$ and $\tau_2$ used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into $\tau_1$ and $\tau_2$ by whether their indices were in the leading equivalence class and reindexing. The $y$ values for $\tau_2$ can be found from the strong coloring’s $Y$ after reversing the index change, and the $y$ values for $\tau_1$ are all 0.

We claim each strong coloring created by this process has the properties described by Definition 13: Because the high-dimension equivalence class $x_{q_1, d-1} = x_{q_2, d-1} = \cdots = x_{q_j, d-1}$ was added at the start of the strong coloring, the high-dimension coefficients of each term are smaller than the low-dimension coefficients. Criterion 1 is satisfied by reindexing the variables. Criterion 2 met by assigning all elements from $\tau_1$ the same $y$ value. The remaining criteria are satisfied because $\tau_1$ and $\tau_2$ satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount strong colorings.

We also claim that every strong coloring of $(\omega^d_{a-j})$ is counted by this process: each can be mapped to some $\tau_1$ and $\tau_2$ that create it by a similar argument to proving that the process creates unique strong colorings. Every strong coloring of $(\omega^d_{a-j})$ must have a leading
equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_1,d-1}$ to satisfy Definition 13 (the equivalence class might only contain one variable); taking only the variables $x_{q,n}$ with indices appearing in that equivalence class (but not those variables in the equivalence class itself) with all-zero $y$ values forms $\tau_1$, a strong coloring for $\binom{\omega^d-1}{j}$. The variables with $q$ indices not in the equivalence class with their $y$ values form $\tau_2$, a strong coloring for $\binom{\omega^d}{a-j}$. The original strong coloring of $\binom{\omega^d}{a}$ was counted by interleaving $\tau_1$ with $\tau_2$ and inserting the leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_1,d-1}$. Therefore the final case of the result holds. 

\[8.1 \quad T(a, \omega^d \cdot k) \geq P(a, \omega^d \cdot k)\]

**Theorem 23.** For integers $a, d, k \geq 0$, $T(a, \omega^d \cdot k) \geq P(a, \omega^d \cdot k)$.

**Proof.** If $P(a, \omega^d \cdot k) = 0$, this is satisfied vacuously because $T(a, \omega^d \cdot k) \geq 0$. Now we can assume $P(a, \omega^d \cdot k) \geq 1$. Let $E = \binom{\omega^d}{a}$. Note that all strong colorings of $E$ are disjoint from each other. That is, for any edge $e \in E$, if $e$ satisfies $\tau'$, then it does not satisfy any nonequivalent strong coloring of $E$. This is because if $e$ were to satisfy two strong colorings $\tau_1$ and $\tau_2$, then $\tau_1$ and $\tau_2$ must share the same $Y$ tuple, equivalence classes and order, so the strong colorings must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\omega^d)}$ and construct a coloring

\[
\text{COL}: E \to [P(a, \omega^d \cdot k)]
\]

\[
e \mapsto \begin{cases} 
 i & \text{e satisfies } \tau_i \\
 1 & \text{otherwise} 
\end{cases}
\]

Similar to Theorem 12, our coloring has two ways to output color 1, both through satisfaction of $\tau_1$ and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of $\tau_1$.

For arbitrary $H \approx \omega^d \cdot k$ and $\tau$, we will find $y_q$ and $z_{qn}$ variables where

\[
\{\omega^d y_1 + \omega^d z_{1,d-1} + \cdots + \omega^1 z_{1,1} + z_{1,0}; \ldots; \omega^d y_a + \omega^d z_{a,d-1} + \cdots + \omega^1 z_{a,1} + z_{a,0}\}
\]

satisfies $\tau$.

Given an arbitrary $H \approx \omega^d \cdot k$ and $\tau$, we first separate $H$ into $k$ ordered sets by the leading coefficient, each order-equivalent to $\omega^d$.

Then, if there are equivalence classes in $\tau$, using the process formally described in the proof of Theorem 18, we consider the leading equivalence class of $\tau$. By criterion 2 of Definition 13, all variables in that equivalence class must come from same set order-equivalent to $\omega^d$. We assign a finite value to that equivalence class, and move to the next class with a potentially different $y$ value, using the assigned finite as a lower bound for the next one. We can repeat this process to find $z_{qn}$ that satisfy every strong coloring of $E$ for arbitrary $H \approx \omega^d$. Then, we can assign

\[
(y_1, y_2, \ldots, y_a) = Y.
\]
By the process described in Theorem 18, the criteria involving \(x\) variables in Definition 13 are satisfied. Because we only assigned \(z\) variables as equal when their elements had equal \(y\) coefficients, criterion 2 is satisfied.

If there are no equivalence classes in \(\tau\), we can simply assign

\[
(y_1, y_2, \ldots, y_a) = Y
\]

to satisfy \(\tau\).

Therefore for all \(H \approx \omega^d\), \(|\text{COL}(\binom{H}{a})| \geq P(a, \omega^d \cdot k)\) so \(T(a, \omega^d \cdot k) \geq P(a, \omega^d \cdot k)\). \(\square\)

8.2 \(T(a, \omega^d \cdot k) \leq P(a, \omega^d \cdot k)\)

**Lemma 24.** For integers \(a, d, k \geq 0\) and \(N \approx \omega\), there exists some \(H \approx \omega^d \cdot k\) where for all \(e \in \binom{H}{a}\), \(e\) satisfies a strong coloring of \(\binom{\omega^d \cdot k}{a}\) and each coefficient in \(e\) is contained in \(N\).

**Proof.** Because \(N \approx \omega\), we can index it \(x_1, x_2, x_3, \ldots\) with \(x_1 < x_2 < x_3 < \cdots\).

If \(d = 0\), \(H' = \{x_1, x_2, \ldots, x_k\}\) suffices.

If \(d \geq 1\), we can first apply Lemma 19 with \(N\) to attain some \(H' \approx \omega^{d+1}\) with the listed properties. Then, let \(H\) be the first \(k\) copies of \(\omega^d\) within \(H'\): formally,

\[
H = \{\omega^d y + \omega^{d-1} x_{d-1} + \ldots + \omega^1 x_1 + x_0 \in H' \mid y < k\}.
\]

Because the edges of \(H'\) satisfied criterion 5 of Definition 13 at dimension \(n = d + 1\), we know the edges of \(H\) satisfy criterion 2. The remaining criteria are satisfied directly because \(H'\) satisfied them. \(\square\)

**Theorem 25.** For integers \(a, d, k \geq 0\), \(T(a, \omega^d) \leq P(a, \omega^d)\).

**Proof.** Let \(E = \binom{\omega^d \cdot k}{a}\) and

\[
\text{COL}: E \rightarrow [b]
\]

be an arbitrary coloring of \(E\) for some finite \(b\).

Enumerate the strong colorings of \(E\) from \(\tau_1\) to \(\tau_{P(a, \omega^d \cdot k)}\). We know the maximum size of any strong coloring of \(E\) is \(a \cdot d\). For each \(\tau_i\), let

\[
 f_i: \binom{\omega}{a \cdot d} \rightarrow E
\]

where if \(\tau_i\) has size \(p\), \(f_i\) maps \(X\) to the unique \(e \in E\) where \(e\) satisfies \(\tau_i\) and the \(p\) equivalence classes of \(e\) are made up of the \(p\) least elements of \(X\). For example, one strong coloring of \(\binom{\omega^2 \cdot 2}{2}\) is

\[
Y = (0, 1), \quad x_{11} < x_{21} < x_{20} < x_{10}.
\]
The corresponding $f_i$ would be

$$f_i : \left( \begin{array}{c} \omega \\ 4 \end{array} \right) \mapsto \left( \begin{array}{c} \omega^2 \cdot 2 \\ 2 \end{array} \right)$$

$$(x_1, x_2, x_3, x_4) \mapsto \text{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_4, \omega^2 \cdot 1 + \omega \cdot x_2 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that the elements of $Y$ are inside of the definition of $f_i$ – the coefficients would be swapped for another strong coloring with identical $x$ permutation but $Y = (1, 0)$.

Then, define

$$\text{COL}' : \left( \begin{array}{c} \omega \\ a \cdot d \end{array} \right) \mapsto \underbrace{[b] \times [b] \times \cdots \times [b]}_{P(a, \omega^d \cdot k) \text{ times}}$$

$$X \mapsto (f_1(X), f_2(X), \ldots, f_{P(a, \omega^d \cdot k)}(X))$$

and apply Theorem 7 to find some $N \approx \omega$ where

$$\text{COL}' \left( \left( \begin{array}{c} N \\ a \cdot d \end{array} \right) \right)$$

expresses only one tuple $Y$ containing $|Y| = P(a, \omega^d \cdot k)$ colors.

Apply Lemma 24 to find some $H \approx \omega^d \cdot k$ with the properties listed in Lemma 24. Now we claim

$$\text{COL} \left( \left( \begin{array}{c} H \\ a \end{array} \right) \right)$$

expresses at most $P(a, \omega^d \cdot k)$ colors.

By Lemma 24, each element $e \in \binom{H}{a}$ satisfies a strong coloring of $E$. Then for some arbitrary edge $e$, let $e$ satisfy $\tau_i$ with size $p \leq a \cdot d$. Then take the $p$ unique values in $e$, and if necessary, insert any new larger natural values from $N$ to form a set of $a \cdot d$ values; denote this $X \in \binom{N}{a \cdot d}$. We know $\text{COL}'(X) = Y$ so by the definition of $\text{COL}'$, $\text{COL}(e) \in Y$. Because $|Y| = P(a, \omega^d \cdot k)$, $T(a, \omega^d \cdot k) \leq P(a, \omega^d \cdot k)$. 

\begin{proof}

By Theorem 23, $T(a, \omega^d \cdot k) \geq P(a, \omega^d \cdot k)$. By Theorem 25, $T(a, \omega^d \cdot k) \leq P(a, \omega^d \cdot k)$. The result follows.
\end{proof}

8.3 $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$

**Theorem 26.** For all $a, d, k \in \mathbb{N}$, $T(a, \omega^d k) = P(a, \omega^d k)$.

**Proof.** By Theorem 23, $T(a, \omega^d \cdot k) \geq P(a, \omega^d \cdot k)$. By Theorem 25, $T(a, \omega^d \cdot k) \leq P(a, \omega^d \cdot k)$. The result follows.
9 $T(a, \alpha)$ for any $\alpha < \omega^\omega$

9.1 Dimensional Strong Colorings

We defined strong colorings to compute Big Ramsey degrees of sets of the form $\omega^d \cdot k$. We’ll now extend the definition to **dimensional strong colorings**, which will allow us to compute Big Ramsey degrees for all ordered sets less than $\omega^\omega$.

**Definition 27.** For an ordered set $\alpha < \omega^\omega$, consider $\alpha$ in terms of a polynomial in $\omega$:

$$\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0.$$  

For some integer $a \geq 0$, we say there are $a$ elements in $\alpha$. Unlike the definition of strong colorings, each element can have anywhere from 0 and $d$ variables, as the element does not have to originate from the $\omega^d$ part of $\alpha$. For $1 \leq q \leq a$, we use $c_q$ for the number of variables element $q$ has (the element therefore originated from the $\omega^{c_q}$ part of $\alpha$). We denote each element as

$$\omega^{c_q} \cdot y_q + \omega^{c_q-1} \cdot x_{q,c_q-1} + \omega^{c_q-2} \cdot x_{q,c_q-2} + \cdots + \omega \cdot x_{q,1} + x_{q,0},$$

where $0 \leq y_q < k_{c_q}$ and each $x_{q,n}$ a nonnegative integer.

A **dimensional strong coloring**, hereafter referred to as a DSC, is first an assignment of the $a c_q$ variables with integers for $0 \leq c_q \leq d$. Then, it’s an assignment of the $a y_q$ variables with integers for $0 \leq y_q < k_{c_q}$. Each element has $x_{q,n}$ variables associated with it for $1 \leq q \leq a$ and $0 \leq n < c_q$. Then, the $x_{q,n}$ variables with $<$ or $=$ signs between them are permuted in a way that satisfies the below criteria. Only criterion 6 is different than the criteria from Definition 13.

1. If $d \geq 1$, $x_{i0} < x_{j0}$ for all $i < j$ (the element indices are ordered by their lowest-dimension variable). If $d = 0$, $y_i < y_j$ for all $i < j$.

2. $y_i \neq y_j \rightarrow x_{in} \neq x_{jn}$ for all $0 \leq n < d$ (Elements that have a different $y$ value have all different $x$ values).

3. $x_{qa} < x_{qb}$ for all $a > b$ (the high-dimension variables of each element are strictly less than the low-dimension variables).

4. $x_{ma} = x_{nb} \rightarrow a = b$ (only variables with the same dimension can be equal).

5. $x_{in} \neq x_{jn} \rightarrow x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$ and $i \neq j$ (elements that differ in a high-dimension variable differ in all lower-dimension variables).

6. $c_i \neq c_j \rightarrow y_i \neq y_j$ and $c_i \neq c_j \rightarrow x_{in} \neq x_{jn}$ for all $0 \leq n < d$ (different $c$ variables mean different $y$ and $x$ values).

**Definition 28.** We again define the **size** of a DSC to be how many equivalence classes its $x$ variables form. A DSC’s size $p$ is still bounded above by $d \cdot a$. 

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Definition 29.

1. $D_p(a, \alpha)$ is the number of strong colorings with size $p$ there are for $\binom{a}{\alpha}$.

2. $D(a, \alpha)$ is the number of strong colorings there are for $\binom{a}{\alpha}$ regardless of size.

Lemma 30. For integers $a, d, k, p \geq 0$, $D_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$.

Proof. Let $\alpha = \omega^d \cdot k$. Because $\alpha$ only has one dimension with nonzero $k$ coefficient, we claim each $c_q = d$: if some $c_q \neq d$, because $y_q < k_c$ by Definition 27, $y_q < 0$, which is impossible. Then there are the same count of $a \cdot d$ $x_{qn}$ variables being permuted, the new criterion 6 doesn’t apply because all $c_q$ are equal. Then both are under the same restrictions so $D_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$. \hfill $\square$

Lemma 31. For all $\alpha < \omega^\omega$ with

$$\alpha \approx \omega^d \cdot k_1 + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0,$$

$$D_p(a, \alpha) = \sum_{j=0}^{a} \sum_{i=0}^{p} \binom{p}{i} P_i(j, \omega^d \cdot k) D_{p-i}(a-j, \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0).$$

when $d > 0$ and $D_p(a, \alpha) = P_p(a, \alpha)$ otherwise.

Proof. When $d = 0$,Lemma 30 shows $D_p(a, \alpha) = P_p(a, \alpha)$. When $d \geq 1$, we will describe a process of combining strong colorings with DSCs to create DSCs for $\binom{a}{\alpha}$.

For arbitrary $a \geq 0, p \geq 0$, and $\alpha < \omega^\omega$, let $0 \leq j \leq a$ and $0 \leq i \leq p$ be integers. We will create

$$\binom{p}{i} P_i(j, \omega^d \cdot k) D_{p-i}(a-j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0)$$

DSCs, with each DSC having $j$ elements from the $\omega^d$ part of $\alpha$ and $a-j$ elements from parts with lower dimensions.

Let $\tau_1$ represent one of the $P_i(j, \omega^d \cdot k)$ strong colorings of $\binom{\omega^d k}{j}$ with size $i$, and $\tau_2$ represent one of the $P_{p-i}(a-j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0)$ DSCs of $\binom{\omega^{d-1} k_{d-1} + \cdots + k_0}{a-j}$ with size $p-i$. We change $\tau_1$ into a DSC by assigning it $c_q = d$ for all $c_q$.

Then we can combine each $\tau_1$ and $\tau_2$ to form $\binom{p}{i}$ unique new DSCs of size $p$. Reindex $\tau_2$ and permute the equivalence classes as in the proof of Lemma 17. Note that we do not insert a leading equivalence class – this is because we are not increasing the dimension or size of $\tau_1$.

We can keep each $y_q$ value the same, and reindex them and the $c_q$ variables as we reindex the $x_{qn}$ variables to ensure criterion 1.

Each DSC is unique by the $\tau_1$ and $\tau_2$ used to create it because the process is invertible: we can identify the elements originally from $\tau_1$ because they uniquely have $c_q = d$.

We claim each DSC created by this process has the properties described by Definition 27: Because all $c_q$ are equal for $\tau_1$, criterion 6 is satisfied for the elements from $\tau_1$. Criterion 1 is
satisfied by reindexing the variables. Criterion 2 met by assigning all elements from \( \tau_2 \) the same \( y \) value. The remaining criteria are satisfied because \( \tau_1 \) and \( \tau_2 \) satisfied them and their internal orders and equivalence classes were preserved in permuting the equivalence classes. Therefore this process does not overcount DSCs.

We also claim that every DSC of \( \binom{a}{\alpha} \) is counted by this process: each can be mapped to some \( \tau_1 \) and \( \tau_2 \) that create it by a similar argument to proving that the process creates unique DSCs.

9.2 \( T(a, \alpha) \leq D(a, \alpha) \)

**Theorem 32.** For all \( \alpha < \omega \),

\[
T(a, \alpha) \leq D(a, \alpha).
\]

**Proof.** If \( D(a, \alpha) = 0 \), this is satisfied vacuously because \( T(a, \alpha) \geq 0 \). Now we can assume \( D(a, \alpha) \geq 1 \). Let \( E = \binom{a}{\alpha} \). Note that all DSCs of \( E \) are disjoint from each other. That is, for any edge \( e \in E \), if \( e \) satisfies \( \tau' \), then it does not satisfy any nonequivalent DSC of \( E \).

This is because if \( e \) were to satisfy two DSC \( \tau_1 \) and \( \tau_2 \), then \( \tau_1 \) and \( \tau_2 \) must share the same equivalence classes, \( y_q, c_q \), and order, so the DSCs must be equivalent. Therefore, we can index them \( \tau_1 \ldots \tau_{P(a, \alpha)} \) and construct a coloring

\[
\text{COL}: E \rightarrow [D(a, \alpha)]
\]

\[
e \mapsto \begin{cases} 
  i & \text{e satisfies } \tau_i \\
  1 & \text{otherwise}
\end{cases}
\]

For arbitrary \( \approx \alpha \) and a DSC for \( \alpha \tau \), we can assign \( c_q \) and \( y_q \) based on \( \tau \). Then, we can apply a similar process to the one used in Theorem 23 to find \( z_{qn} \) variables that match the permutation of \( x_{qn} \) variables.

Let \( \approx \alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0 \). We can separate \( H \) into \( d + 1 \) sets order-equivalent to \( \omega^n \cdot k_n \), and separate each of those into \( k_n \) sets order-equivalent to \( \omega^n \).

Then, for each equivalence class in \( \tau \), using the process formally described in the proof of Theorem 18, we consider the leading equivalence class of \( \tau \). By criteria 2 and 6 of Definition 27, all variables in that equivalence class must come from same set order-equivalent to \( \omega^n \).

We assign a finite value to that equivalence class, and move to the next class with potentially different \( c \) and \( y \) values, using the assigned finite as a lower bound for the next one. We can repeat this process to find \( z_{qn} \) that satisfy each DSC of \( E \) for arbitrary \( \approx \alpha \). By the process described in Theorem 18, the criteria involving \( x \) variables in Definition 27 are satisfied. Because we only assigned \( z \) variables as equal when their elements had equal \( y \) coefficients, criterion 2 is satisfied. By the same logic, criterion 6 is satisfied.

Therefore for all \( \approx \alpha \), \( |\text{COL}(\binom{H}{a})| \geq D(a, \alpha) \) so \( T(a, \alpha) \geq D(a, \alpha) \).

9.3 \( T(a, \alpha) \geq D(a, \alpha) \)

**Lemma 33.** For \( \alpha < \omega^\omega \), integers \( a \geq 0 \), and \( N \approx \omega \), there exists some \( \approx H \approx \alpha \) where for all \( e \in \binom{H}{a} \), e satisfies a DSC of \( \binom{a}{\alpha} \) and each coefficient in \( e \) is contained in \( N \).
Proof. Because $N \approx \omega$, we can index it $x_1, x_2, x_3, \ldots$ with $x_1 < x_2 < x_3 < \cdots$. Let $\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0$.

First apply Lemma 24 on $N$ to produce an $H' \approx \omega \cdot (d + 1)$. For $0 \leq n \leq d$, let $H'_n \approx \omega$ such that

$$H' = \sum_{n=0}^{d} H'_n.$$ 

For $0 \leq n \leq d$, apply Lemma 24 on $H'_n$ to yield some $H_n \approx \omega^n \cdot k_n$ where all $e \in H_n$ satisfy a strong coloring. Then let

$$H = \sum_{n=0}^{d} H_n$$

so that $H \approx \alpha$.

Because all $e \in H_n$ satisfy a strong coloring for $0 \leq n \leq d$, only criterion 6 of Definition 27 remains to be satisfied. Since we separated $N$ into disjoint orders $H'_n$, we know each $H_n$ is disjoint so criterion 6 is satisfied.

Theorem 34. For all $\alpha < \omega^\omega$,

$$T(a, \alpha) \geq D(a, \alpha).$$

Proof. Let $E = \binom{\alpha}{d}$ and

$$\text{COL}: E \to [b]$$

be an arbitrary coloring of $E$ for some finite $b$.

Enumerate the DSCs of $E$ from $\tau_1$ to $\tau_{D(a, \alpha)}$. We know the maximum size of any DSC of $E$ is $a \cdot d$. For each $\tau_i$, let

$$f_i : \binom{\omega}{a \cdot d} \to E$$

where if $\tau_i$ has size $p$, $f_i$ maps $X$ to the unique $e \in E$ where $e$ satisfies $\tau_i$ and the $p$ equivalence classes of $e$ are made up of the $p$ least elements of $X$. For example, one DSC of $\binom{\omega^2 + \omega \cdot 8}{2}$ is

$$c_1 = 2, c_2 = 1, y_1 = 0, y_2 = 6, \; x_{11} < x_{20} < x_{10}.$$ 

The corresponding $f_i$ would be

$$f_i : \binom{\omega}{4} \to \binom{\omega^2 + \omega \cdot 8}{2}$$

$$(x_1, x_2, x_3, x_4) \mapsto \text{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_3, \omega \cdot 6 + x_2\})$$

where $x_1 < x_2 < x_3 < x_4$. 

Then, define

\[
\text{COL'} : \left( \frac{\omega}{a \cdot d} \right) \to [b] \times [b] \times \cdots \times [b] \quad \text{times}
\]

\[
X \mapsto (f_1(X), f_2(X), \ldots, f_{D(a, \alpha)}(X))
\]

and apply Theorem 7 to find some \( N \approx \omega \) where

\[
\text{COL'} \left( \left( \frac{N}{a \cdot d} \right) \right)
\]

expresses only one tuple \( Y \) containing \( |Y| = D(a, \alpha) \) colors.

Apply Lemma 33 to find some \( H \approx \alpha \) with the properties listed in Lemma 33. Now we claim

\[
\text{COL} \left( \left( \frac{H}{a} \right) \right)
\]

expresses at most \( D(a, \alpha) \) colors.

By Lemma 33, each element \( e \in \binom{H}{a} \) satisfies a DSC of \( E \). Then for some arbitrary edge \( e \), let \( e \) satisfy \( \tau_i \) with size \( p \leq a \cdot d \). Then take the \( p \) unique values in \( e \), and if necessary, insert any new larger natural values from \( N \) to form a set of \( a \cdot d \) values; denote this \( X \in \binom{N}{a \cdot d} \). We know \( \text{COL'}(X) = Y \) so by the definition of \( \text{COL'} \), \( \text{COL}(e) \in Y \). Because \( |Y| = D(a, \alpha) \), \( T(a, \omega^d \cdot k) \leq D(a, \alpha) \).

\[ \Box \]

9.4 \( T(a, \alpha) = D(a, \alpha) \)

**Theorem 35.** For all \( \alpha < \omega \),

\[
T(a, \alpha) = D(a, \alpha).
\]

**Proof.** By Theorem 34, \( T(a, \alpha) \geq D(a, \alpha) \). By Theorem 32, \( T(a, \alpha) \leq D(a, \alpha) \). The result follows. \( \Box \)

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**References**

References


