ON SETS OF DISTANCES OF n POINTS

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1. The function f(n). Let $[P_n]$ be the class of all planar subsets P_n of n points and denote by $f(n)$ the minimum number of different distances determined by its n points for P_n an element of $\{P_n\}$. Clearly, $f(3)=1$ (with the three points forming the vertices of an equilateral triangle) $f(4)=2$, $f(5)=2$. The following theorem establishes rough bounds for arbitrary n. Though I have sought to improve this result for many years, I have not been able to do so.

THEOREM 1. *The minimum number $f(n)$ of distances determined by n points of a plane satisfies the inequalities*

$$(n-3/4)^{1/2}-1/2 \leqq f(n) \leqq cn/(\log n)^{1/2}.$$

Proof. Let P_1 be an arbitrary vertex of the least convex polygon determined by the n points, and denote by K the number of different distances occurring among the distances P_1P_i $(i=2, 3, \cdots, n)$. If N is the maximum number of times the same distance occurs, then clearly $KN \geqq n-1$.

If r is a distance that occurs N times then there are N points on the circle with center P_1 and radius r, which all lie on the same semi-circle (since P_1 is a vertex of the least convex polygon). Denoting these points by $Q_1, Q_2, \cdots, Q_N$, we have $Q_1Q_2 < Q_1Q_3 < \cdots < Q_1Q_N$, and these $N-1$ distances are pairwise distinct. Thus $f(n) \geqq \max(N-1, (n-1)/N)$, which is a minimum when $N(N-1)=n-1$. This yields the first part of the theorem.

Considering now the points (x, y) with *integer* coordinates for $0 \leqq x, y \leqq n^{1/2}$, we obtain at least n points P_i which pairwise have distances of the form $(u^2+v^2)^{1/2}$, $0 \leqq u \leqq n^{1/2}$, $0 \leqq v \leqq n^{1/2}$. Now it is well-known that the number of different integers not exceeding $2n$ which are of the form u^2+v^2 is less than $cn/(\log n)^{1/2}$, and the proof is complete.*

For n points in k-dimensional space the same method yields $c_1n^{1/k} < f(n) < c_2n^{2/k}$.

2. Some conjectures concerning f(n). Let us assume that our n points form a convex polygon. Then I conjecture that $f(n) \geqq [n/2]$, with the equality sign valid when the n points are vertices of a regular n-gon. I am unfortunately unable to prove this. The following conjecture is stronger: In every convex polygon there is at least one vertex with the property that no three vertices of the polygon are equally distant from it. If this is the case, then clearly we would obtain $[n/2]$ different distances by considering all the distances from such a vertex.

A still stronger conjecture is that on every convex curve there exists a point P such that every circle with center P intersects the curve in at most 2 points.

3. The function g(n; r). Denoting by $g(n; r)$ the maximum number of times a given distance r can occur among n points of a plane we establish

* Landau, Verteilung der Primzahlen, vol. 2.

THEOREM 2. $n^{1+c/\log\log n} < g(n;r) < n^{3/2}$.

Proof. Assuming that there are x_i points at distance r from P_i, clearly $g(n;r) = \max \frac{1}{2}\sum_{i=1}^{n} x_i$. We suppose that $x_1 \geqq x_2 \geqq \cdots \geqq x_n$. Now the x_i points at distance r from P_i can contain at most two points with distance r from P_j. Hence

$$\sum_{i=1}^{j} (x_i - 2i + 2) \leqq n \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{1}$$

Put $[n^{1/2}] = a$, $n^{1/2} - a = \epsilon$, $0 \leqq \epsilon < 1$. We have from (1)

$$\begin{aligned} x_1 + x_2 + \cdots + x_a \leqq n + 2\binom{a}{2} &= 2n - 2\epsilon n^{1/2} + \epsilon^2 - n^{1/2} + \epsilon \\ &< 2n - 2\epsilon n^{1/2} \end{aligned} \tag{2}$$

for $n \geqq 4$. Thus

$$x_a < \frac{1}{a}(2n - 2\epsilon n^{1/2}) = 2n^{1/2}. \tag{3}$$

Hence from (2) and (3)

$$\sum_{i=1}^{n} x_i < 2n - 2\epsilon n^{1/2} + (n - a)2n^{1/2} = 2n^{3/2}$$

or

$$g(n; r) < n^{3/2}.$$

By again considering the set of points (x, y), $0 \leqq x$, $y \leqq a$ we easily obtain (using well known theorems about the number of solutions of $u^2 + v^2 = m$)*

$$g(n) > n^{1+c/\log\log n}$$

which completes the proof.

It seems likely that $g(n) < n^{1+\epsilon}$.

4. Maximum and minimum distances. If r is the diameter of the points P_i, it is well known that r can occur only n times.† This follows almost immediately from the fact that if $\overline{P_1P_2} = r$ and $\overline{P_3P_4} = r$ the lines P_1P_2 and P_3P_4 must intersect, for otherwise a simple argument shows that the diameter of $P_1P_2P_3P_4$ would be greater than r. Connect P_i with P_j if and only if their distance is r. We distinguish two cases. In Case 1, every P_i is connected with at most two other P's. In this case the number of lines, *i.e.*, of pairs of points at distance r is clearly $\leqq n$.

* See *e.g.* P. Erdös, London Math. Soc. Journal, 1937, vol. 12, p. 133. The proof would depend on the prime number theorem for primes of the form $4k+1$ (or on some weaker elementary result concerning the distribution of primes of the form $4k+1$).

† Jahresbericht der Deutschen Math. Vereinigung, vol. 43, 1934, p. 114.

If P_1 would be connected with three vertices say P_2, P_3, P_4 where P_1P_3 is between P_1P_2 and P_1P_4 then P_3 can not be connected with any other P_i, since P_3P_i would have to intersect both P_1P_2 and P_1P_4 (the angle $P_2P_1P_4$ is of course $\leqq\pi/3$), and thus be greater than r. Now we can just omit P_3 and since both the number of points and the number of distances are reduced by 1, the proof can be completed by induction.

It would be interesting to have an analogous result for n points in k dimensional space. Vázsonyi* conjectured that in three-dimensional space the maximum distance can not occur more than $2n-2$ times.

If one could prove that in k-dimensional space the maximum distance can not occur more than kn times, the following conjecture of Borsuk would be established: *Each k-dimensional* subset of diameter 1 can be decomposed into $k+1$ summands each having diameter <1.

Let now r' denote the minimal distance between any two P's. First it is easy to see that r' can not occur more often than $3n$ times. This is immediately clear from the fact that since r' was the minimal distance between any two P's, there can be no more than 6 P's at distance r' from any given P.

Connect P_i with P_j if and only if their distance is r'. A simple argument shows that no two such lines P_1P_2 and P_3P_4 can intersect (otherwise there would be two P's at distance $<r'$). Thus the graph we obtain is planar, and from Euler's theorem it follows that the number of edges of such a graph is not greater than $3n-6$. Thus we have proved the following

THEOREM 3. *Let the maximum and minimum distances determined by n points in a plane be denoted by r and r', respectively. Then r can occur at most n times and r' at most* $3n-6$ *times.*

It is easy to give n points where the maximum distance occurs exactly n times. By more complicated arguments we can prove that the minimal distance r' can occur not more than $3n-cn^{1/2}$ times, where c is a constant. On the other hand the example of the triangular lattice shows that r' can occur $3n-c_1n^{1/2}$ times. I did not succeed in determining exactly how often r' can occur.

One could try to generalize Theorem 3 to higher dimensions. But already the case of three-dimensional space presents great difficulties. It would be of some interest to determine the maximum number of points on the unit sphere of k dimensions such that the distance of any two is $\geqq 1$.

* Oral communication.