Upper Bounds on the Large Ramsey Number $\mathrm{LR}_2(k)$
Exposition by William Gasarch (gasarch@cs.umd.edu)

1 Introduction

Paris and Harrington [2] proved *The Large Ramsey Theorem.* We will examine the case of Large Ramsey for graphs (stated below). The general case is interesting in that the associated function grows so quickly that the proof is not in Peano Arithmetic. This was Paris and Harrington's motivation (see Appendix). We give upper bounds in the case of 2-coloring the edges of a graph. The graph case is provable in Peano Arithmetic. None of this manuscript is original.

Notation 1.1

1. Let $k, n \in \mathsf{N}$, $k < n$. $K_{[k,n]}$ is the complete graph on the vertices $\{k, k+1, \ldots, n\}$ (This is *not* the complete bipartite graph with k vertices on the left and n vertices on the left even though the notation looks similar.)

2. Let K_ω be the complete graph on the vertices of N.

3. Let $K_{[k,\omega)}$ be the complete graph on the vertices $\{k, k+1, \ldots\}$.

4. We will only be coloring EDGES of complete graphs. Henceforth in this manuscript the term *coloring G* will mean coloring the *edges* of G.

5. Assume that a complete graph (on a finite or infinite number of vertices) is colored. A *homogeneous set* is a set of vertices of the graph such that every edge between them has the same color. A set is *homogeneous RED* if it is homogeneous and the color is RED (similar for BLUE).

6. Let $A \subseteq \mathsf{N}$. A is *large* if A is larger than its minimal element.

   Example 1.2

   (a) The set $\{10, 15, 20, \ldots, 100\}$ is large since it has $20 \geq 10$ elements.
   (b) The set $\{10^{10}, 10^{10}+1, \ldots, 10^{10}+10^9\}$ is not large since it has $10^9 + 1 < 10^{10}$ elements.

7. The notation μx means *least* x. The domain is assumed to be the natural numbers. For example
$$f(y) = \mu x[x^2 \geq y]$$
would be another way to express $f(y) = \lceil \sqrt{y} \rceil$.

Recall the infinite Ramsey Theorem:

Theorem 1.3 *For every $c \in \mathsf{N}$, for all c-colorings of K_ω, there exists an infinite homogeneous set.*

We use this to give a proof of the Large Ramsey Theorem:

Theorem 1.4 *Let $c \in \mathsf{N}$. For all k there exists n such that for every c-coloring of $K_{[k,n]}$ there exists a large homogeneous set.*

Proof:

Assume, by way of contradiction, that the theorem is false. Let c, k be such that for all $n > k$ there exists a c-coloring of $K_{[k,n]}$, which we denote COL_n, such that there is no large homogeneous set relative to COL_n. We use the COL_n's to create a coloring of $K_{[k,\omega)}$ which we call COL.

List out the edges of $K_{[k,\omega)}$: $e_1, e_2, \ldots$. We color each edge as follows.

Initially set

$$\begin{aligned} \mathrm{COLORINGS}_1 = &\ \ \mathsf{N} \\ \mathrm{COL}(e_1) = &\ \ \mu d[\exists^\infty n \in \mathrm{COLORINGS}_1 \text{ such that } \mathrm{COL}_n(e_1) = d] \end{aligned}$$

Let $i \geq 2$. Assume inductively that

1. $e_1, \ldots, e_{i-1}$ are colored.
2. $\mathrm{COLORINGS}_i = \{j : \mathrm{COL}_j(e_1) = \mathrm{COL}(e_1), \ldots, \mathrm{COL}_j(e_{i-1}) = \mathrm{COL}(e_{i-1})\}$.
3. $\mathrm{COLORINGS}_i$ is infinite.

Let

$$\begin{aligned} \mathrm{COL}(e_i) = &\ \ \mu d[\exists^\infty n \in \mathrm{COLORINGS}_i \text{ such that } \mathrm{COL}_n(e_i) = d] \\ \mathrm{COLORINGS}_{i+1} = &\ \ \mathrm{COLORINGS}_i \cap \{n : \mathrm{COL}_n(e_i) = d\} \end{aligned}$$

It is easy to see that, after each step, the conditions $1, 2, 3$ still hold.

COL is a c-coloring of $K_{[k,\omega)}$. By Theorem 1.3 there is an infinite homogeneous set

$$H = \{f_1 < f_2 < f_3 < \cdots\}.$$

Take the first f_1 vertices,

$$H' = \{f_1 < f_2 < f_3 < \cdots < f_{f_1}\}.$$

This is a homogeneous set relative to COL. By the definition of COL there is at least one (in fact infinitely many) n such that COL_n agrees with COL on all of the edges between elements of H'. Hence H' is a large homogeneous subset relative to COL_n. This contradicts the definition of COL_n. ∎

Definition 1.5 Let $\mathrm{LR}_c(k)$ be the least n such that n satisfies the conclusion of Theorem 1.4.

Note that the proof gave no bounds on $\mathrm{LR}_c(k)$. In this manuscript we do the following:

We give a proof of Theorem 1.4 in the $c = 2$ case that yields the following: for all $k \geq 3$, $\mathrm{LR}_2(k) \leq 2^{k^{2k}}$. The proof is due to Erdos and Mills [1]. Later Mills improved this to $\mathrm{LR}_2(k) \leq 2^{2^{2.942k}}$. (We do not include the proof of the improved result.)

2 $\mathrm{LR}_2(k) \le (k+1)^{((k-1)!)^2}$

Theorem 2.1 *For $k \ge 3$ the following hold.*

1. $\mathrm{LR}_2(k) \le (k+1)^{((k-1)!)^2}$

2. $\mathrm{LR}_2(k) \le 2^{k^{2k}}$ *(this follows from part 1 and algebra).*

Proof:

We set $n = (k+1)^{((k-1)!)^2}$, however, this will not come into play until the very end.

Assume, by way of contradiction, that there is a 2-coloring of $K_{[k,n]}$ with no large homogeneous set. Let COL be that coloring. We define two sequences of vertices. This will look similar to one of the usual proofs of (the ordinary) Ramsey's Theorem; however, we will point out the differences.

Let

1. $a_0 = k$.

2. For all $i \ge 1$

$$a_i = \mu x[a_{i-1} < x \le n \text{ and } \{a_0, \ldots, a_{i-1}, x\} \text{ is Homogeneous RED}].$$

   Note that a_i might not exist. If a_i does not exist then, for all $i' > i$, $a_{i'}$ does not exist.

3. $b_0 = k$.

4. For all $i \ge 1$

$$b_i = \mu x[b_{i-1} < x \le n \text{ and } \{b_0, \ldots, b_{i-1}, x\} \text{ is Homogeneous BLUE}].$$

   Note that b_i might not exist. If b_i does not exist then, for all $i' > i$, $b_{i'}$ does not exist.

Note 2.2

1. If $a_0, a_1, \ldots, a_{k-1}$ are defined then we have our large homogeneous set. If $b_0, b_1, \ldots, b_{k-1}$ are defined then we have our large homogeneous set. Hence we assume they both stop before they get to $k-1$. We also assume that they both stop at the same place (A similar proof works if they stop at different places.) Let L be the maximum index such that a_L and b_L are defined. We assume that a_{L+1}, b_{L+1} do not exist. We also assume $L \le k-2$.

2. We are not tossing out any vertices as is common in proofs of Ramsey's theorem. However, we will keep track of the vertices that are not in either sequence very carefully.

Claim 1: $a_0 = b_0 = k$. For all $i \ge 1$, for all $j \ge 0$ $a_i \ne b_j$ and $b_i \ne a_j$.
Proof of Claim 1:

Clearly $a_0 = b_0 = k$. For all $i, j \ge 1$ $a_i > a_0 = b_0$ and $b_j > b_0 = a_0$. Hence we need only consider $i, j \ge 1$.

If $i, j \ge 1$ then $\mathrm{COL}(a_0, a_i) = RED$ and $\mathrm{COL}(b_0, b_j) = \mathrm{COL}(a_0, b_j) = BLUE$. Hence $a_i \ne b_j$.
End of Proof of Claim 1

We now carefully partition the elements that were *not* chosen to be in either sequence.

Motivation: Lets say that

$$a_0 < a_1 < \cdots < a_\ell$$

are all defined. Let x be such that

$$x > a_\ell \text{ and } x \notin \{a_{\ell+1}, \ldots, a_L\} \cup \{b_{\ell+1}, \ldots, b_L\}.$$

Assume that $\mathrm{COL}(a_0, x) = RED$.

Why is $x \notin \{a_{\ell+1}, \ldots, a_L\}$?

There are several possibilities (though this list is not exhaustive):

1. $\mathrm{COL}(a_0, x) = RED$ but $\mathrm{COL}(a_1, x) = BLUE$.
2. $\mathrm{COL}(a_0, x) = \mathrm{COL}(a_1, x) = RED$, but $\mathrm{COL}(a_2, x) = BLUE$.

$\vdots$

ℓ-1. $\mathrm{COL}(a_0, x) = \cdots = \mathrm{COL}(a_{\ell-2}, x) = RED$ but $\mathrm{COL}(a_{\ell-1}, x) = BLUE$.

ℓ. $\mathrm{COL}(a_0, x) = \cdots = \mathrm{COL}(a_{\ell-1}, x) = RED$ but $\mathrm{COL}(a_\ell, x) = BLUE$.

Assume that $\mathrm{COL}(a_0, x) = BLUE$.

Why is $x \notin \{b_{\ell+1}, \ldots, b_L\}$?

There are several possibilities (though this list is not exhaustive):

1. $\mathrm{COL}(b_0, x) = BLUE$ but $\mathrm{COL}(b_1, x) = RED$.
2. $\mathrm{COL}(b_0, x) = BLUE = \mathrm{COL}(b_1, x) = BLUE$, but $\mathrm{COL}(b_2, x) = RED$.

$\vdots$

ℓ-1. $\mathrm{COL}(b_0, x) = \cdots = \mathrm{COL}(b_{\ell-2}, x) = BLUE$ but $\mathrm{COL}(b_{\ell-1}, x) = RED$.

ℓ. $\mathrm{COL}(b_0, x) = \cdots = \mathrm{COL}(b_{\ell-1}, x) = BLUE$ but $\mathrm{COL}(b_\ell, x) = RED$.

End of Motivation

We now define sets of x's that are not in $\{a_{\ell+1}, \ldots, a_L, b_{\ell+1}, \ldots, b_L\}$ as motivated above.

Definition 2.3 For $1 \le \ell \le L$ define the following sets.

$$A_\ell = \{x : a_\ell < x \le n \wedge \{a_0, \ldots, a_{\ell-1}, x\} \text{ is Homogeneous RED } \wedge \mathrm{COL}(a_\ell, x) = BLUE\}.$$

$$B_i = \{x : b_\ell < x \le n \wedge \{b_0, \ldots, b_{\ell-1}, x\} \text{ is Homogeneous BLUE } \wedge \mathrm{COL}(b_\ell, x) = RED\}.$$

Claim 2:

1. $\{k,\ldots,n\} - \{a_0,\ldots,a_L,b_1,\ldots,b_L\} = A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L$.
2. For all $1 \le \ell, \ell' \le L$, $A_\ell \cap B_{\ell'} = \emptyset$.
3. For all $1 \le \ell < \ell' \le L$, $A_\ell \cap A_{\ell'} = \emptyset$ and $B_\ell \cap B_{\ell'} = \emptyset$.
4. $n+1 = k + (2L+1) + |A_1| + \cdots |A_L| + |B_1| + \cdots + |B_L|$. (We will later see why we chose to bound $n+1$ instead of n and why we put the $2L+1$ in parenthesis. We do not really need the equality. Just having the $\le$ is all we need. Hence we do not even need parts 2 and 3. But parts 2 and 3 and the equality are good to know.)

Proof of Claim 2:
1) Let $x \in \{k,\ldots,n\} - \{a_0,\ldots,a_L,b_1,\ldots,b_L\}$. Recall that $a_0 = b_0$. There are two cases.

Case RED**:** $\mathrm{COL}(a_0,x) = \mathrm{COL}(b_0,x) = RED$

Let j be the largest number such that $a_j < x$. Why was x not chosen to be a_{j+1}? Since

$$\mathrm{COL}(a_0,x) = RED$$

there is an ℓ, $1 \le \ell \le j$ such that

$$\mathrm{COL}(a_0,x) = \mathrm{COL}(a_1,x) = \cdots = \mathrm{COL}(a_{\ell-1},x) = RED$$

but

$$\mathrm{COL}(a_\ell,x) = BLUE.$$

Hence $x \in A_\ell$. Since $1 \le \ell \le j \le L$ we have

$$x \in A_1 \cup \cdots \cup A_L$$

Case $BLUE$**:** $\mathrm{COL}(a_0,x) = \mathrm{COL}(b_0,x) = BLUE$ By similar reasoning we have

$$x \in B_1 \cup \cdots \cup B_L$$

Combining Case RED and Case $BLUE$ we get

$$\{k,\ldots,n\} - \{a_0,\ldots,a_L,b_1,\ldots,b_L\} \subseteq A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L.$$

We now prove the reverse inclusion. Assume $x \in A_\ell$ (the case of $x \in B_\ell$ is similar).

- $a_\ell < x \le n$, hence $x \in \{k,\ldots,n\} - \{a_0,\ldots,a_\ell\}$.
- $\mathrm{COL}(a_\ell,x) = BLUE$, hence $x \notin \{a_{\ell+1},\ldots,a_L\}$.
- $\mathrm{COL}(a_\ell,x) = RED$ hence $x \notin \{b_0,\ldots,b_L\}$.

Putting this all together we get $x \in \{k, \ldots, n\} - \{a_0, \ldots, a_L, b_1, \ldots, b_L\}$. Hence

$$A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L \subseteq \{k, \ldots, n\} - \{a_0, \ldots, a_L, b_1, \ldots, b_L\}.$$

2) If $x \in A_\ell$ then $\text{COL}(a_0, x) = RED$. If $x \in B_{\ell'}$ then $\text{COL}(a_0, x) = BLUE$. Hence $A_\ell \cap B_{\ell'} = \emptyset$.

3) If $x \in A_\ell$ then $\text{COL}(a_\ell, x) = BLUE$. If $\ell < \ell'$ and $x \in A_{\ell'}$ then $\text{COL}(a_\ell, x) = RED$. Hence $A_\ell \cap A_{\ell'} = \emptyset$.
4) By part 1:

$$\{k, \ldots, n\} - \{a_0, \ldots, a_L, b_1, \ldots, b_L\} = A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L.$$

$$|\{k, \ldots, n\} - \{a_0, \ldots, a_L, b_1, \ldots, b_L\}| = |A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L|.$$

Since $n + 1 \notin \{a_0, \ldots, a_L, b_1, \ldots, b_L\}$ and not in $A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L$ we have

$$|\{k, \ldots, n+1\} - \{a_0, \ldots, a_L, b_1, \ldots, b_L\}| = |A_1 \cup \cdots \cup A_L \cup B_1 \cup \cdots \cup B_L| + 1.$$

By parts 2 and 3 all of the A's and B's are disjoint. Hence

$$n + 1 - (k - 1) - (2L + 1) = 1 + |A_1| + \cdots + |A_L| + |B_1| + \cdots + |B_L|.$$

$$n + 1 = 1 + (k - 1) + (2L + 1) + |A_1| + \cdots + |A_L| + |B_1| + \cdots |B_L|.$$

$$n + 1 = k + (2L + 1) + |A_1| + \cdots |A_L| + |B_1| + \cdots + |B_L|.$$

End of Proof of Claim 2

How big can A_ℓ and B_ℓ be? We bound it using the ordinary Ramsey Function.

Definition 2.4 $R(k_1, k_2)$ is the least number n such that for all 2-colorings of K_n there is either a RED homogeneous set of size k_1 or a BLUE homogeneous set of size k_2. $R(k_1, k_2)$ exists by the ordinary Ramsey's theorem.

Claim 3:

1. For $1 \le \ell \le L$, $|A_\ell| < R(k - \ell, a_\ell - 1)$
2. For $1 \le \ell \le L$, $|B_\ell| < R(k - \ell, b_\ell - 1)$

Proof of Claim 3:

We prove part 1. Part 2 is similar. Assume, by way of contradiction, that $|A_\ell| \ge R(k-\ell, a_\ell-1)$. Apply the ordinary Ramsey theorem to the induced complete graph on the vertices of A_ℓ.
Case 1: There is a RED homogeneous set of size $k - i$. Call this set H. Recall that for every element of $x \in A_\ell$ (and hence every $x \in H$)

$$\text{COL}(a_0, x) = \text{COL}(a_1, x) = \cdots = \text{COL}(a_{\ell-1}, x) = RED.$$

Hence the set

$$H \cup \{a_0, \ldots, a_{\ell-1}\}$$

is homogeneous RED. It is of size $(k - \ell) + i = k$ and its least element is $a_0 = k$. Hence it is a large homogeneous set. This contradicts the initial assumption about COL that it has no large homogeneous sets.

Case 2: There is a BLUE homogeneous set of size $a_\ell - 1$. Call this set H. Recall that for every element $x \in A_\ell$ (and hence every $x \in H$)

$$\text{COL}(a_\ell, x) = BLUE.$$

Hence the set

$$H \cup \{a_\ell\}$$

is homogeneous BLUE. It is of size $(a_\ell - 1) + 1 = a_\ell$ and its least element is a_ℓ (recall that all elements of A_ℓ are $> a_\ell$). Hence it is a large homogeneous set. This contradicts the initial assumption about COL that it has no large homogeneous sets.

End of Proof of Claim 3

You might think OH, WE ARE DONE since, by Claim 2,

$$n + 1 = k + (2L + 1) + |A_1| + \cdots + |A_L| + |B_1| + \cdots |B_L|.$$

and by Claim 3 we have bounds on $|A_\ell|$ and $|B_\ell|$. But alas, here is what we can get from this information:

$$\begin{aligned} n+1 &= k + (2L+1) + |A_1| + \cdots + |A_L| + |B_1| + \cdots |B_L| \\ &\leq 3k + 1 + R(k-1, a_1 - 1) + \cdots + R(k-L, a_L - 1) + R(k-1, b_1 - 1) + \cdots + R(k-L, b_L - 1) \\ &\leq 3k + 1 + 2L \times R(k-1, \max\{a_L, b_L\} - 1) \\ &\leq 3k + 1 + 2k \times R(k-1, \max\{a_L, b_L\} - 1) \end{aligned}$$

What bounds do we have on a_L, b_L? All we know is that $a_L, b_L \leq n$. So we obtain

$$n + 1 \leq 3k + 1 + 2k \times R(k - 1, n - 1).$$

This is not helpful at all! :-(.

Lets look back to our bound on $n + 1$:

$$n + 1 = k + (2L + 1) + |A_1| + \cdots + |A_L| + |B_1| + \cdots + |B_L|.$$

This should be useful! There should be a way to bound this! As the saying goes *sometimes its easier to solve a harder problem.* Lets try to bound partial sums. We will first bound a_i's and b_i's by sums of the A's and B's. These bounds will look similar to the bound on $n + 1$ and the proofs will be similar. We will then bound these partial sums. But we need to know the order of the a_i's and b_i's.

List out

$$a_0, a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, b_L$$

in order and rename them:

$$k = c_0 < c_1 < c_2 < \cdots < c_{2L}.$$

By Claim 1 these are all distinct.

For $1 \le i \le 2L$ let

$$C_i = \begin{cases} A_j & \text{if } c_i = a_j \\ B_j & \text{if } c_i = b_j. \end{cases}$$

We define $c_{2L+1} = n + 1$. By Claim 2 we have

$$c_{2L+1} \le k + (2L+1) + |C_1| + \cdots + |C_{2L}|.$$

(We can now say why we stated the original inequality on $n+1$ as we did. We used $n+1$ instead of n since n could be one of the a_i's or b_i's. We put the $2L+1$ in parenthesis so it matched the index of the c when we defined $c_{2L+1} = n+1$.)

We will show, by a proof similar to Claim 2, that, for all $1 \le i \le 2L$

$$c_i \le k + i + |C_1| + \cdots + |C_{i-1}|.$$

This does not seem like progress. We want to bound the sum, not bound something else by the sum, or by a partial sum. But trust me, it will be useful for our final goal.

Claim 4:

1. $\{k, \ldots, c_i - 1\} - \{c_0, \ldots, c_{i-1}\} \subseteq C_1 \cup \cdots \cup C_{i-1}$. (A better but clumsier statement is true. We will note it but not use it.)
2. $c_i \le k + i + |C_1| + \cdots + |C_{i-1}|$.

Proof of Claim 4

1)

We assume that $c_i = a_{j_1}$ and that the largest index of a b-term is j_2. The proof is similar if the roles of a and b are reversed. We use α as an index if the index does not matter. Do not assume that the α are all the same. We are assuming that

$$c_0 < c_1 < \cdots < c_\alpha = b_{j_2} < a_\alpha < \cdots < a_{j_1} = c_i.$$

Let $x \in \{k, \ldots, c_i - 1\} - \{c_0, \ldots, c_{i-1}\}$. Recall that $c_0 = a_0 = b_0$. There are two cases. (These two cases are similar to those in the proof of Claim 2.)

Case RED**:** $\mathrm{COL}(a_0, x) = \mathrm{COL}(b_0, x) = RED$. Let j be the largest number such that $a_j < x$. Why was x not chosen to be a_{j+1}? Since

$$\mathrm{COL}(a_0, x) = RED$$

there is an ℓ, $1 \le \ell \le j$ such that

$$\mathrm{COL}(a_0, x) = \mathrm{COL}(a_1, x) = \cdots = \mathrm{COL}(a_{\ell-1}, x) = RED$$

but

$$\mathrm{COL}(a_\ell, x) = BLUE.$$

Hence $x \in A_\ell$. Since $1 \le \ell \le j$ we have

$$x \in A_1 \cup \cdots \cup A_j.$$

In Claim 2 we had the obvious fact that $j \le L$. Here we have something stronger. What are the bounds on j? Since

$$x \in \{k, \ldots, c_i - 1\} = \{k, \ldots, a_{j_1} - 1\}$$

and $a_j < x$ hence $j < j_1$. Hence

$$x \in A_1 \cup \cdots \cup A_{j_1 - 1}.$$

Case $BLUE$**:** $\mathrm{COL}(a_0, x) = \mathrm{COL}(b_0, x) = BLUE$ By similar reasoning

$$x \in B_1 \cup \cdots \cup B_{j_2 - 1}.$$

Combining Case RED and Case $BLUE$ we have

$$\{k, \ldots, c_i - 1\} - \{c_0, \ldots, c_{i-1}\} \subseteq A_1 \cup \cdots \cup A_{j_1 - 1} \cup B_1 \cup \cdots \cup B_{j_2 - 1}.$$

Notice that the right hand side is actually a subset of $C_1 \cup \cdots \cup C_{i-1}$. It is missing B_{j_2} which is one of the C's. This is the clumsier statement we alluded to earlier. However, we add that C value back in and have

$$\{k, \ldots, c_i - 1\} - \{c_0, \ldots, c_{i-1}\} \subseteq C_1 \cup \cdots \cup C_{i-1}.$$

2) By part 1 we have

$$\{k, k+1, \ldots, c_i - 1\} - \{c_0, c_1, \ldots, c_{i-1}\} \subseteq C_1 \cup \cdots \cup C_{i-1}.$$

$$|\{k, k+1, \ldots, c_i - 1\} - \{c_0, c_1, \ldots, c_{i-1}| \le |C_1 \cup \cdots \cup C_{i-1}|.$$

$$(c_i - 1) - (k - 1) - i \le |C_1| + \cdots + |C_{i-1}|.$$

$$c_i - k - i \le |C_1| + \cdots + |C_{i-1}|.$$

$$c_i \le k + i + |C_1| + \cdots + |C_{i-1}|.$$

End of Proof of Claim 4

We have c_i bounded by a sum of $|C_j|$ where $j < i$. We have bounds on $|C_j|$ from Claim 3. Can we use this to get bounds on c_i? NO. We will try and see where it fails.

Since $C_j \in \{A_1, \ldots, A_j, B_1, \ldots, B_j\}$ we have

$$\begin{aligned}
|C_j| \le\ & \max\{|A_1|,\ldots,|A_j|,|B_1|,\ldots,|B_j|\}\\
\le\ & \max\{R(k-1,a_1-1),R(k-2,a_2-1),\ldots,R(k-j,a_j-1),R(k-1,b_1-1,\ldots,R(k-j,b_j-1)\}\\
\le\ & R(k-1,\max\{a_j,b_j-1\})
\end{aligned}$$

Lets say that the sequence

$$c_0 < c_1 < \cdots < c_{2j}$$

is

$$a_0 < b_1 < a_1 < b_2 < \cdots < a_j < b_j.$$

Then $a_j = c_{2j-1}$ and $b_j = c_{2j}$. Hence we obtain

$$|C_j| \le R(k-1, c_{2j}-1)$$

$$c_i \le k+i+|C_1|+\cdots+|C_{i-1}| \le k+i+(i-1)R(k-1,c_{2i-2}).$$

This is NOT helpful since we are bounding c_i by a term that involves c_{2i-2}.

We need to bound, for all i, $k+i+|C_1|+\cdots+|C_{i-1}|$. We will do this. The bound on c_i from Claim 4 will be used in the proof.

We need to define a recurrence tailor made for our application. First we define a sequence of bits.

Definition 2.5 We define a sequence $\sigma \in \{0,1\}^{2L}$. For $1 \le i \le 2L$

$$\sigma(i) = \begin{cases} 0 & \text{if } c_i \text{ is some } a_j \ ; \\ 1 & \text{if } c_i \text{ is some } b_j \end{cases} \qquad (1)$$

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2L)$. Note that σ has L 0's and L 1's and that $L \le k-1$.

Definition 2.6 We define f on strings with $\le k-1$ 0's and $\le k-1$ 1's as follows.

1. $f(\lambda) = k+1$
2. $f(\tau 0) = f(\tau) + R(k-j, f(\tau)-1)$ where j is the number of 0's in τ.
3. $f(\tau 1) = f(\tau) + R(k-j, f(\tau)-1)$ where j is the number of 1's in τ.

Notation 2.7 If $\tau \in \{0,1\}^*$ then $\tau[i..j]$ is $\tau(i)\cdots\tau(j)$.

Claim 5: For all $1 \le i \le 2L$

$$k + i + |C_1| + \cdots + |C_{i-1}| \le f(\sigma[1..i-1])$$

Proof of Claim 5

We prove this by induction on i.

Base case: For $i = 1$ we need

$$k + 1 + |C_1| + \cdots + |C_0| \le f(\lambda)$$

Since $|C_1| + \cdots + |C_0| = 0$ and $f(\lambda) = k + 1$, this is true.

Induction Hypothesis (IH): Claim 5 is true for i. Note what this gives us:

$$k + i + |C_1| + \cdots + |C_{i-1}| \le f(\sigma[1..i-1])$$

We prove Claim 5 for $i + 1$.

$$\begin{aligned} k + (i+1) + |C_1| + \cdots + |C_i| &= (k + i + |C_1| + \cdots + |C_{i-1}|) + |C_i| + 1; \\ &\le f(\sigma[1..i-1]) + |C_i| + 1 \text{ by IH }. \end{aligned}$$

We assume $c_i = a_j$ (the case of $c_i = b_j$ is similar). This also gives us that $\sigma(i) = 0$. KEY: j is the number of 0's in $\sigma[1..i]$. Note that $C_i = A_j$ and $|A_j| < R(k-j, a_j - 1)$. Hence

$$|C_i| + 1 \le R(k-j, a_j - 1) = R(k-j, c_i - 1).$$

Now we need a bound on c_i. OH, we have one from Claim 4! Recall that Claim 4 states

$$c_i \le k + i + |C_1| + \cdots |C_{i-1}|.$$

WOW- the induction hypothesis states

$$k + i + |C_1| + \cdots + |C_{i-1}| \le f(\sigma[1..i-1])$$

Hence we have

$$c_i \le k + i + |C_1| + \cdots + |C_{i-1}| \le f(\sigma[1..i-1])$$

We use this to obtain

$$|C_i| + 1 \le R(k-j, a_j - 1) = R(k-j, c_i - 1) \le R(k-j, f(\sigma[1..i-1]) - 1).$$

Therefore

$$f(\sigma[1..i-1]) + |C_i| + 1 \le f(\sigma[1..i-1]) + R(k-j, f(\sigma[1..i-1]) - 1).$$

By the definition of f we have

$$f(\sigma[1..i]) = f(\sigma[1..i-1] + R(k-j, f(\sigma[1..i-1]) - 1.$$

Hence we have our result.

End of Proof of Claim 5

To bound the recurrence we need the following fact from Ramsey Theory

Fact 2.8 *If $2 \le k_1 < k_2$ then*
$$R(k_1, k_2) \le k_2^{k_1-1} - k_2^{k_1-2}.$$

Proof of Fact

This follows from the well known inequality
$$R(a, b) \le R(a, b-1) + R(a-1, b).$$

End of proof of Fact

Definition 2.9 If $\tau \in \{0,1\}^*$ then $numzeros(\tau)$ is the number of zero's in τ and $numones(\tau)$ is the number of ones in τ.

Claim 6: Let $k \in \mathsf{N}$, $k \ge 3$, and $\sigma \in \{0,1\}^*$. Assume that $numzeros(\sigma), numones(\sigma) \le k-1$. Then
$$f(\sigma) \le (k+1)^{\frac{(k-1)!^2}{(k-numzeros(\sigma)-1)!(k-numones(\sigma)-1)!}}.$$

Proof of Claim 6:

We prove this by induction on $numzeros(\sigma) + numones(\sigma)$

Base Case: $numzeros(\sigma) + numones(\sigma) = 0$

Then $\sigma = \lambda$ and we have
$$f(\sigma) \le (k+1)^{\frac{(k-1)!^2}{(k-1)!(k-1)!}} = k+1.$$

Induction Hypothesis (IH): Let σ be the string we want to prove Claim 6 for. We can assume that for all τ with $numzeros(\tau) + numones(\tau) < numzeros(\sigma) + numones(\sigma)$, Claim 6 holds.

We can assume $numzeros(\sigma), numones(\sigma) \le k-1$. We assume $\sigma = \tau 0$ (the case where $\sigma = \tau 1$ is similar) Clearly $numzeros(\tau) = numzeros(\sigma) - 1$ and $numones(\tau) = numones(\sigma)$. Hence we can apply the IH to τ. Also note that, since $numzeros(\sigma) \le k-1$, $numzeros(\tau) \le k-2$.

By the definition of f and Fact 2.8

$$f(\tau 0) = f(\tau) + R(k - numzeros(\tau), f(\tau) - 1) \le f(\tau) + f(\tau)^{k-numzeros(\tau)-1} - f(\tau)^{k-numzeros(\tau)-2}.$$

We break into cases. Our main concern is how much the $-f(\tau)^{k-numzeros(\tau)-2}$ term will help counter the $f(\tau)$ term.

1. If $k - numzeros(\tau) - 2 \ge 1$ then $f(\tau) - f(\tau)^{k-numzeros(\tau)-2} \le 0$ hence both terms can be ignored. Using this and the induction hypothesis we get

$$f(\tau 0) \le f(\tau)^{k-numzeros(\tau)-1} \le ((k+1)^{\frac{((k-1)!)^2}{(k-numzeros(\tau)-1)!(k-numones(\tau)-1)!}})^{k-numzeros(\tau)-1}$$

$$\le (k+1)^{\frac{((k-1)!)^2}{(k-numzeros(\tau)-2)!(k-numones(\tau)-1)!}}$$

$$= (k+1)^{\frac{((k-1)!)^2}{(k-numzeros(\sigma)-1)!(k-numones(\sigma)-1)!}}$$

(This happens when $numzeros(\tau) \le k-3$. Since $numzeros(\tau) \le k-2$ the only case left is $numzeros(\tau) = k-2$.)

2. If $numzeros(\tau) = k-2$ then

$$f(\tau 0) \le f(\tau) + f(\tau)^{k-numzeros(\tau)-1} - f(\tau)^{k-numzeros(\tau)-2} = f(\tau) + f(\tau) - 1.$$

By the induction hypothesis, algebra, and $k \ge 3$, one can show that

$$f(\tau 0) \le (k+1)^{\frac{((k-1)!)^2}{(k-numzeros(\sigma)-1)!(k-numones(\sigma)-1)!}}.$$

End of Proof of Claim 6

By Claim 2 we have

$$n+1 \le k + (2L+1) + |C_1| + \cdots + |C_{2L}|.$$

Let σ be as defined in Definition 2.5. By Claim 5

$$k + (2L+1) + |C_1| + \cdots + |C_{2L}| \le f(\sigma).$$

By Claim 6

$$f(\sigma) \le (k+1)^{(\frac{(k-1)!}{(k-L-1)!})^2}.$$

Using $L \le k-2$ and the last three inequalities we obtain

$$n+1 \le (k+1)^{((k-1)!)^2}.$$

This contradicts the definition of n.

∎

3 Appendix

Notation 3.1

1. Let $k, n \in \mathsf{N}$, $k < n$. $K^m_{[k,n]}$ is the complete m-hypergraph on the vertices $\{k, k+1, \ldots, n\}$ (This looks like the notation for the cross product of graphs or an m-partite graph, but its not.)
2. Let K^m_ω be the complete m-hyperegraph on the vertices of N.
3. Let $K^m_{[k,\omega)}$ be the complete m-hypergraph on the vertices $\{k, k+1, \ldots\}$.
4. We will only be coloring EDGES of complete m-hypergraphs. Henceforth in this manuscript the term *coloring* G will mean coloring the *edges* of G.

5. Assume that a complete m-hypergraph (on a finite or infinite number of vertices) is colored. A *homogeneous set* is a set of vertices of the graph such that every edge between them has the same color.

6. Let $A \subseteq \mathsf{N}$. A is *large* if A is larger than its minimal element. (Same as in main paper.)

Recall the infinite hypergraph Ramsey Theorem:

Theorem 3.2 *For every $m \in \mathsf{N}$, for every $c \in \mathsf{N}$, for all c-colorings of K_ω^m, there exists an infinite homogeneous set.*

This can be used to give a proof of the Large Ramsey Theorem:

Theorem 3.3 *For every $m \in \mathsf{N}$, for every $c \in \mathsf{N}$, for all k there exists n such that for every c-coloring of $K_{[k,n]}^m$ there exists a large homogeneous set.*

We omit the proof, though it is similar to the proof of Theorem 1.3.

Definition 3.4 Let $LR_c^m(k)$ be the n in Theorem 3.3.

Paris and Harrington showed that Theorem 3.3 cannot be proven in Peano Arithmetic. They showed that the function $LR_c^m(k)$ grows faster than any function whose existence can be proven in Peano Arithmetic. This essentially means that the proof from the Theorem 3.2 is really the only proof- so to prove this finitary theorem requires infinitary techniques. So a proof like that of the original finite Ramsey Theorem, or of the bound in $\mathrm{LR}_2(k)$ in this manuscript, cannot be obtained for Theorem 3.3.

References

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