Open Sets are Ramsey
Exposition by William Gasarch

1 Introduction

Notation 1.1 Let $a \in \mathsf{N}$. $\binom{\mathsf{N}}{a}$ is the set of all a-sized subsets of N. $\binom{\mathsf{N}}{\omega}$ is the set of all infinite subsets of N.

Def 1.2 Let $a \in \mathsf{N}$. Let COL be a 2-coloring of $\binom{\mathsf{N}}{a}$. $H \in \binom{\mathsf{N}}{\omega}$ is *homogeneous* if there exists a color c such that every $X \in \binom{H}{a}$ is colored c. Note that we built into the definition of homogenous that the set is infinite.

Recall the infinite a-ary Ramsey's theorem (the 2-color case)

Theorem 1.3 *For all $a \in \mathsf{N}$, for all 2-colorings of $\binom{\mathsf{N}}{a}$, there exists a homogenous set.*

Why stop at $a \in \mathsf{N}$? What about the case of $a = \omega$?
IS the following true:
Theorem? For all 2-colorings of $\binom{\mathsf{N}}{\omega}$, there exists an infinite homogenous set.

NO. In Section 3 we provide a counterexample. But what if the coloring is *well behaved*?

Def 1.4

1. A 2-coloring of $\binom{\mathsf{N}}{\omega}$ is *Ramsey* if there exists a homogenous set.
2. A subset RED of $\binom{\mathsf{N}}{\omega}$ is *Ramsey* if the coloring formed by coloring all elements of RED, RED, and coloring all other sets $BLUE$, is homogenous.
3. A subset $\mathcal{X}$ of $2^{\binom{\mathsf{N}}{\omega}}$ is called *Ramsey* if every $A \in \mathcal{X}$ is Ramsey.

Convention 1.5 We identify elements of $\binom{\mathsf{N}}{\omega}$ with elements of $\{0,1\}^\omega$ that have an infinite number of 1's.

Def 1.6 Let RED be a subset of $\binom{\mathsf{N}}{\omega}$.

1. If $\sigma \in \{0,1\}^*$ then
$$O_\sigma = \{A \mid \sigma \preceq A \wedge (\exists^\infty i)[A[i] = 1]\}.$$
2. RED is a *Basic Open Set* if it is of the form O_σ. We denote the set of basic open sets by Σ_0 or Π_0.

3. *RED* is *Open* if it is a finite or countable union of basic open sets. We denote the set of open sets by Σ_1. Note that the open sets here are the same induced by the metric $d(x,y) = \frac{1}{1+i}$ where i is the least number x and y differ on.

4. *RED* is *Closed* if it is the finite or countable intersection of basic open sets. We denote the set of open sets by Π_1. It is easy to show that a set if Closed iff its complement is open.

5. We define Σ_α, Π_α, and Δ_α inductively for all countable ordinals α.

   - A set is in Σ_α if it is the finite or countable union of sets in $\Pi_{<\alpha}$.
   - A set is in Π_α if it is the finite or countable intersection of sets in $\Sigma_{<\alpha}$.
   - A set is Δ_α if it is in both Σ_α and Π_α.
   - A set is *Borel* if there exists an α such that it is in Σ_α or Π_α.

Are their elements of $\binom{\mathsf{N}}{\omega}$ that are not Borel? Yes, the set of Borel sets is countable. Is there a nice class of sets that are just barely not Borel? Yes.

Def 1.7 Let *GREEN* be a subset of $\binom{\mathsf{N}}{\omega} \times \binom{\mathsf{N}}{\omega}$.

1. *GREEN* is a *Basic Open Set* if it is the set of all pairs of strings with an infinite number of 1's in $\sigma_1\{0,1\}^\omega \times \sigma_2\{0,1\}^\omega$. We denote the set of basic open sets by Σ_0 or Π_0.

2. Σ_α, Π_α, Δ_α, and Borel subsets of $\binom{\mathsf{N}}{\omega} \times \binom{\mathsf{N}}{\omega}$ are defined similar to Definition 1.6.

3. Let *RED* be a subset of $\binom{\mathsf{N}}{\omega}$. *RED* is *analytic* if there exists a Green Borel set $GREEN \subseteq \binom{\mathsf{N}}{\omega} \times \binom{\mathsf{N}}{\omega}$ such that

$$RED = \{X \mid (\exists Y)[(X,Y) \in GREEN]\}.$$

It is known that there are analytic sets that are not Borel.

In Section 4 we show that open sets are Ramsey. Galvin and Prikry [2] first proved this. Actually they proved something stronger: that every Borel Set is Ramsey. More is known: Silver [5] showed that all Analytic sets are Ramsey. Ellentuck [1] and Tanaka [7] have alternative proofs.

2 Examples

We will break with mathematical tradition and give some examples.

Example 2.1

1.
$$RED = \{x \mid x(1)x(2)x(3)x(4)x(5) = 00110\}.$$
RED is a basic open set. RED is Ramsey since the following is a $BLUE$-homoegenous set:
$$\{x \mid x(1)x(2)x(3)x(4)x(5) = 00000\}.$$
There are no RED-homogenous sets.

2.
$$RED = \{x \mid (\exists i)[i \in x \wedge\ i \text{ is prime }]\}.$$
RED is an open set as it is the union over all primes p of
$$\{x \mid 0^{p-1}1 \preceq x\}.$$
RED is Ramsey:

- The following set is RED-homogenous: THE PRIMES.
- The following set is $BLUE$-homogenous: THE NONPRIMES.

(Prime can be replaced with any infinite subset of N and everything said here still holds.)

3.
$$RED = \{x \mid (\forall i)[i \in x \implies i \text{ is prime }]\}.$$
Complement is
$$\{x \mid (\exists i)[i \in x \implies i \text{ is not prime }]\}.$$
By the above item $\overline{RED}$ is open, so RED is closed.

4.
$$RED = \{x \mid (\exists^{\infty} i)[i \in x \wedge\ i \text{ is prime }]\}.$$
First let
$$RED_n = \{x \mid (\exists \text{ at least } ni)[i \in x \wedge\ i \text{ is prime }]\}.$$
RED_n is open as it is the union as σ has $\ge n$ primes in it of O_σ. So RED_n is Σ_1. Hence the intersection of all of the RED_n is Π_2. This is RED. RED is Ramsey:

- The following set is RED-homogenous: THE PRIMES.
- The following set is $BLUE$-homogenous: THE NONPRIMES.

(Prime can be replaced with any infinite subset of N and everything said here still holds.)

5. Let $x \in \{0,1\}^\omega$. Let $RED = \{x\}$. So RED is a singleton set. RED is closed (or Π_1) since
$$RED = \bigcap_{\sigma \preceq x} O_\sigma.$$
RED is Ramsey- let H be any proper infinite subset of x. It is a $BLUE$-homoegenous set. There are no RED-homogenous sets.

6. Let RED be a countable union of elements of $\{0,1\}^*$. Using the above item RED is a countable union of closed sets (Π_1 sets) and hence RED is Σ_2. One can show that RED is Ramsey by constructing a $BLUE$ homogenous set via diagonziation.

7.
$$RED = \{x \mid (\exists^\infty i)[x[i] = 1 \wedge (\forall 1 \le j \le i)[x[i+j] = 0] \wedge x[2i+1] = 1\}.$$
So there are an infinitely number of blocks that have $x[i] = 1$ and then have i 0's and then a 1. Let
$$RED_n = \{x \mid (\exists \text{ at least } n\ i)[x[i] = 1 \wedge (\forall 1 \le j \le i)[x[i+j] = 0] \wedge x[2i+1] = 1\}.$$
RED_n is open. RED is the intersection of the RED_n and hence is Π_2.

3 The Counter Example

We first do a counterexample to a different theorem that is instructive.

Theorem 3.1 *There exists a 2-coloring of $\binom{\mathsf{N}}{2}$ that has no computable homogenous set. (Note that we do not keep track of how complicated the 2-coloring is.)*

Proof:
List out all of the computable 0-1 valued on one variable:
$$f_1, f_2, \ldots,$$
We construct a coloring COL to satisfy the following requirements:

For all $e < \omega$:
R_e: if $(\exists^\infty i)[f_e(i) = 1]$ then f_e is not an indicator function for a homog set.

We color in stages. By state $e < \omega$ we will color at most $2e$ pairs.
Construction:
Stage e. We satisfy R_e. If $(\exists^\infty w)[f_e(w) = 1]$ then do the following, else goto the next stage. Let x, y, z be natural numbers such that

- $f_e(x) = f_e(y) = f_e(z) = 1$,
- $COL(x, y)$ is not yet defined, and
- $COL(x, z)$ is not yet defined.

(We later show that such exists.) We define $COL(x, y) = RED$ and $COL(x, z) = BLUE$. Now f_e cannot be the indicator function for a homogenous set.

We need to show that such x, y, z exist. Let

$$W = \{i \mid f_e(i) = 1\}.$$

KEY: By state e we will have only defined COL on a finite number of elements of $\binom{\mathsf{N}}{2}$ (at most $2e$). Since W is infinite there will exist $x, y, z \in W$ such that $COL(x, y)$, $COL(x, z)$ are not defined. Hence x, y, z exist.

State ω: For all x, y such that $COL(x, y)$ is not defined, we define it to be RED.

End of Construction

▮

By comments made in the proof, the construction works.

The KEY to the proof was that there are only a countable number of requirements and, at every stage $e < \omega$, only a finite number of edges have been colored. The proof above would work with any countable number of functions you want to not have as indicator functions for homogenous sets. What if there is an uncountable number of functions that you want to not have as indicator functions of homogenous sets? The proof would not go through since by stage α where α is uncountable you would have already colored all of the edges. This is a good thing— if the proof did go through you would have disproved Ramsey's Theorem.

Note 3.2 Specker [6] (see also [3, 4] has shown that there is a computable coloring with no computable homogenous set.

Theorem 3.3 *There exists a 2-coloring of* $\binom{\mathsf{N}}{\omega}$ *that has no homogenous set.*

Proof:

List out all of the infinite subsets of N via their indicator functions.

$$f_1, f_2, \ldots,$$

The $\ldots$ is tricky. Let α be the least ordinal of cardinality $2^{\aleph_0}$. We index the f_e by all ordinals $< \alpha$ (this requires the Axiom of Choice). Note that for any $\beta < \alpha$ there are LESS THAN $2^{\aleph_0}$ functions in the set $\{f_\gamma \mid \gamma \leq \beta\}$.

We construct a coloring COL to satisfy the following requirements:

For all $\gamma < \alpha$

R_γ: if $(\exists^\infty i)[f_\gamma(i) = 1]$ then f_γ is not an indicator function for a homog set.

Stage γ. We satisfy R_γ. If $(\exists^\infty w)[f_\gamma(w) = 1]$ then do the following, else goto the next stage. Let $X, Y, Z \in \binom{\mathsf{N}}{\omega}$ be disjoint sets such that

- for all $x \in X$, $y \in Y$, $z \in Z$, $f_\gamma(x) = f_\gamma(y) = f_\gamma(z) = 1$,
- $COL(X \cup Y)$ is not yet defined, and
- $COL(X \cup Z)$ is not yet defined.

(We later show that such X, Y, Z exists.)

We define $COL(X \cup Y) = RED$ and $COL(X \cup Z) = BLUE$. Now f_γ cannot be the indicator function for a homogenous set.

We need to show that such X, Y, Z exist. Let

$$W = \{i \mid f_\gamma(i) = 1\}.$$

KEY: By state γ we will have only defined COL on $< 2^{\aleph_0}$ elements of $\binom{\mathsf{N}}{\omega}$. Since W is infinite there will exist $X, Y \subseteq^\omega W$ such that $COL(X \cup Y)$, $COL(X \cup Z)$ are not defined. Hence X, Y, Z exist.

State α: For all X such that $COL(X)$ is not defined, we define it to be RED.

End of Construction

▮

Note 3.4 The proof of Theorem 3.3 used the Axiom of Choice. What happens if the Axiom of Determinacy is used instead? Is the theorem false? Are all sets Ramsey? We believe these questions are open.

4 Ramsey's Theorem is True for Open Colorings

Our goal is to show that if COL is a 2-coloring of $\binom{\mathsf{N}}{\omega}$ where RED is an open set then there exists an infinite homogenous set.

For this section *coloring* means a 2-coloring of $\binom{\mathsf{N}}{\omega}$. We won't need to use that RED is open until the very end.

Def 4.1 Let COL be a coloring. Let X be a finite subset of N. Let M, N be an infinite subset of N.

1. $X < M$ means that $\max(X) < \min(M)$. So every element of M is bigger than every element of X.
2. $N \subseteq^\omega M$ means that N is an infinite subset of M.

3. M *loves* X if, for all $N \subseteq^\omega M - X$, $COL(X \cup N) = RED$.
4. M *hates* X if $(\forall N \subseteq^\omega M - X)(\exists N' \subseteq^\omega N)[COL(X \cup N') = BLUE]$.

Note 4.2 It is quite possible to have an M and an X such that M neither loves nor hates X.

Lemma 4.3 *Let COL be a coloring. If M loves X then any subset of M loves X. If M hates X then any subset of M hates X.*

Examples

1. RED = all sets that contain all of the evens.

   $M = \{x \mid s \geq 100 \wedge x \text{ is prime}\}$.

   $X = \{2, 4, 6, 8\}$.

   Does M love X? NO: $N = M - \{101\}$ is a subset of $M - X$ such that $N \cup X$ is $BLUE$.

   Does M hate X? YES: If $N \subseteq^\omega M - X$ then $N \cup X$ is $BLUE$.

   Note: M hates $\emptyset$.

2. RED = all sets that contain all of the evens.

   $M = \{x \mid x \equiv 0 \pmod 2 \wedge x \geq 100\}$.

   $X = \{2, 4, 6, 8, \ldots, 98, 100, 102, 104\}$.

   Does M love X? NO: $N = M - \{100, 102, 104, 106\}$ is a subset of $M - X$ such that $N \cup X$ is $BLUE$.

   Does M hate X? YES: every proper $N \subseteq^\omega M - X$ will have $N \cup X$ $BLUE$.

   Note: M hates $\emptyset$.

3. RED = all sets that contain an infinite number of evens

   $M = \{x \mid x \equiv 0 \pmod 2 \wedge x \geq 100\}$.

   $X = \{2, 4, 6, 8, \ldots, 98, 100, 102, 104\}$.

   Does M love X? YES: if $N \subseteq^\omega M - X$ then $X \cup X$ will have an infinite number of evens, so its RED.

   Note: M loves $\emptyset$.

4. RED = all sets that contain an infinite number of evens

   $M = \{x \mid x \equiv 0 \pmod 2 \wedge x \geq 100\}$.

   $X = \{1, 3, 5, 6\}$.

   Does M love X? YES: if $N \subseteq^{\omega} M - X$ then $X \cup X$ will have an infinite number of evens, so its RED.

   Note: M loves $\emptyset$.

5. RED = all sets that contain an infinite number of evens

   $M = \{x \mid x \equiv 0 \pmod 2 \wedge x \geq 100\} \cup \{x \mid x \equiv 0 \pmod 3 \wedge x \geq 100\}$.

   $X = \{2, 4, 6, 8, \ldots, 98, 100, 102, 104\}$.

   Does M love X? NO: Take N to be all numbers ≥ 100 that are $\equiv 3 \pmod 6$. $N \subseteq^{\omega} ODD$ so N is BLUE. This set has all odds in it so its BLUE.

   Does M hate X? NO: Take N to be all numbers ≥ 100 that are $\equiv 0 \pmod 2$. All subsets of this set are RED.

   Note: M has a subset that loves the empty set: the set of evens ≥ 100.

6. RED = all sets such that if a number is IN then the successor is OUT.

   $M = \{x \mid x \equiv 0 \pmod 2\}$.

   $X = \{1, 3, 9\}$.

   M loves X: Its KEY that we are taking $N \subseteq^{\omega} M - X$ so we can't have $2 \in N$ and hence cannot have $1, 2 \in X \cup N$.

   Note: M loves $\emptyset$.

7. RED = all sets that contain an infinite number of evens

   $M = \{x \mid x \equiv 0 \pmod 2 \wedge x \geq 100\} \cup \{x \mid x \equiv 0 \pmod 3 \wedge x \geq 100\}$.

   $X = \{2, 3\}$.

   M does NOT love X since every set that contains X is $BLUE$

   M hates X.

End of Examples

For the rest of this section we fix a coloring COL. Love and Hate will be defined relative to COL. We will only in the very last theorem put a condition on COL.

Lemma 4.4 *Let X be finite. For all $M > X$, M infinite,*

- *there exists $M' \subseteq^{\omega} M - X$ such that M' loves X, or*
- *there exists $M' \subseteq^{\omega} M - X$ such that M' hates X.*

Proof: There are two cases.

1. There exists $N \subseteq^\omega M - X$ such that for all $M'' \subseteq^\omega N$, $COL(X \cup M'') = RED$. Then just $M' = N$. Note that this case can be written as

$$(\exists N \subseteq^\omega M - X)(\forall M'' \subseteq^\omega N)[COL(X \cup M'') = RED].$$

2. For all $N \subseteq^\omega M - X$ there exists $M'' \subseteq^\omega N$ such that $COL(X \cup M'') = BLUE$. This is the very definition of M hates X so take $M' = M$.

▮

Lemma 4.5 *There exists an infinite set M such that, for all finite $X \subseteq M$, either M loves X or M hates X.*

Proof:

If M is an infinite set and X is a finite set then let $L(M, X)$ be the infinite set that results from applying Lemma 4.4.

$$\begin{aligned} M_0 &= \mathsf{N} \\ M_1 &= L(M_0, \emptyset) \\ a_1 &= \text{the least element of } M_1 \\ M_2 &= L(M_1, \{a_1\}) \\ a_2 &= \text{the least element of } M_2 \end{aligned}$$

We need to do the L-operator to BOTH $\{a_1, a_2\}$ and $\{a_2\}$. (We do not need to do the L-operator with $\{a_1\}$ because all later subsets already love or hate $\{a_1\}$ by Lemma 4.3.) We will not define a new a_3 until we are done. To avoid too much notation we will reuse M_3. We write this in a way so it will generalize.

$$\begin{aligned} M_3 &= M_2 \\ M_3 &= L(M_3, \emptyset \cup \{a_2\}) \\ M_3 &= L(M_3, \{a_1\} \cup \{a_2\}) \\ a_3 &= \text{the least element of } M_3 \end{aligned}$$

Now the general case. Assume that $M_{n-1}, a_1, \ldots, a_{n-1}, a_n$ are defined. Let $Y_1, \ldots, Y_{2^{n-1}}$ be all subsets of $\{a_1, \ldots, a_{n-1}\}$.

$$\begin{aligned} M_n &= M_{n-1} \\ M_n &= L(M_n, Y_1 \cup \{a_n\}) \\ M_n &= L(M_n, Y_2 \cup \{a_n\}) \\ \vdots &= \vdots \\ M_n &= L(M_n, Y_{2^{n-1}} \cup \{a_n\}) \\ a_{n+1} &= \text{the least element of } M_n \end{aligned}$$

Let $M = \{a_1, a_2, a_3, \ldots\}$. We show that this works. Let X be a finite subset of M. Let a_n be the max element of X. When M_n was defined M_n either hated of loved X. Note that $M \subseteq^\omega M_n$. By Lemma 4.3 M will either love or hate X. ▮

Lemma 4.6 *Let M be the set from Lemma 4.5. Let X be finite such that $X \subseteq M$. If M hates X then, for almost all $n \in M$, M hates $X \cup \{n\}$.*

Proof: Let

$$N = \{n \mid n \in M \wedge M \text{ loves } X \cup \{n\}\}.$$

We show that N is finite. Since for all n M either hates or loves $X \cup \{n\}$ we obtain the lemma.

Assume, by way of contradiction, that N is infinite. Note that $N \subseteq^\omega M$. We show that EVERY $N' \subseteq^\omega N$ is such that $COL(X \cup N') = RED$. This will contradict M hating X.

Let n be the least element of N' and $N'' = N' - \{n\}$. Note that $X \cup N' = (X \cup \{n\}) \cup N''$ Since $n \in N$, M loves $X \cup \{n\}$. Hence

$$COL(X \cup N') = COL((X \cup \{n\}) \cup N'') = RED.$$

▮

Lemma 4.7 *Let M be the set from Lemma 4.5. If M hates $\emptyset$ then there exists $N \subseteq^\omega M$ such that N hates every finite subset of itself.*

Proof: Assume that M hates $\emptyset$. By Lemma 4.6 there exists $a_1 \in M$ such that M hates $\{a_1\}$.

Assume that $a_1, \ldots, a_{n-1} \in M$ are defined such that M hates all subsets of $\{a_1, \ldots, a_{n-1}\}$. By Lemma 4.6 (applied 2^{n-1} times) there exists $a_n \in M$ such that M hates all subsets of $\{a_1, \ldots, a_n\}$.

Let $N = \{a_1, a_2, \ldots, \}$. Since M hates every finite subset of N, and $N \subseteq M$, by Lemma 4.3 N hates every subset of N. ▮

Theorem 4.8 *If RED is an open set then there exists an infinite homogenous set.*

Proof:

Let M be the set constructed in Lemma 4.5. If M loves $\emptyset$ then M is a homogenous set (since every subset of it is RED). If not, then by Lemma 4.7, there exists a infinite set N that hates every subset of itself.

We show that every infinite subset of N is $BLUE$. Assume, by way of contradiction, that some $N' \subseteq^\omega N$ was RED. Since the RED is an open set there exists σ such that

- σ is an initial segment of N'
- Every set with initial segment σ is RED. Hence every subset of N' that includes σ is RED.

Let X be the finite set coded by σ. Every subset of N' that includes X is RED. This contradicts N hating X. ▮

5 Are there any Interesting Examples?

In this section we look at particular open sets (which we will call RED) and see what the homogenous set is. None of them seem to need the full strength of the proof of the theorem. We would like to see either an interesting open set that uses the full strength of the theorem, or an easier proof.

Let $A \subseteq \{0,1\}^*$ and

$$RED = \bigcup_{\sigma \in A} \sigma\{0,1\}^{\omega}.$$

We look at a variety of types of sets A.

Examples

1) $(\exists n)(\forall \sigma \in A)[\sigma(n) = 1]$.

So every RED set has n. Just let The set $H = \mathsf{N} - \{n\}$ is homogenous $BLUE$. There are no RED homogeneous sets: If H is an infinite subsets then $H - \{n\}$ is $BLUE$.

2) $(\forall \sigma \in A)(\exists i \in \mathsf{N})[\sigma(i)\sigma(i+1) = 11]$.

So every RED set has two consecutive numbers in it. The set $H = EVEN$ is homogenous $BLUE$. There are no RED homogeneous sets: If H is an infinite subsets then one can carefully remove elements so that there are never two in a row (but there are still an infinite number).

3) $(\exists X \subseteq \mathsf{N})(\forall \sigma \in A)(\exists i \in X)[0^i 1 \preceq \sigma]$.

As an example let X be the primes. We are saying that every set in A has as its first elements some primes. The set $H = X$ is homogenous RED. No matter how many elements you remove it will still have as its first element some element of X The set $H = \overline{X}$ is homogenous $BLUE$ for the same reason.

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